Question

Liquid-Drop Model: The coulomb energy of Z protons uniformly distributed throughout a spherical nucleus of radius R is given by

$$E_C = \frac{3}{5}\frac{Z(Z-1)e^2}{4\pi\epsilon_0 R}$$

(a) On the assumption that the mass difference ΔM between a pair of mirror isobars is entirely due to the difference Δm between the ${}^1_1\text{H}$ and neutron masses and to the difference between their coulomb energies, derive a formula for R in terms of ΔM, Δm and Z, where Z is the atomic number of the nucleus with the smaller number of protons. (b) Use this formula to find the radii of the mirror isobars ${}^{15}_{7}\text{N}$ and ${}^{15}_{8}\text{O}$.

Solution

(a) Difference in coulomb energy between mirror isobars ${}^{A}_{Z}\text{X}$ and ${}^{A}_{Z+1}\text{Y}$

$$\begin{aligned}\Delta E_C &= \frac{3}{5}\frac{e^2}{4\pi\epsilon_0 R}\Big((Z+1)Z - Z(Z-1)\Big)\\ &= \frac{3Ze^2}{10\pi\epsilon_0 R}\end{aligned}$$

The coulomb energy of ${}^{A}_{Z+1}\text{Y}$ is clearly higher, but its mass is smaller since a neutron has been replaced by a proton, hence

$$\begin{aligned}\Delta Mc^2 &= \frac{3Ze^2}{10\pi\epsilon_0 R} - \Delta mc^2\\ (\Delta M + \Delta m)c^2 &= \frac{3Ze^2}{10\pi\epsilon_0 R}\\ R &= \frac{3Ze^2}{10\pi\epsilon_0(\Delta M + \Delta m)c^2}\end{aligned}$$

(b) For the mirror isobars ${}^{15}_{7}\text{N}$ and ${}^{15}_{8}\text{O}$ (using atomic masses in the Appendix)

$$\Delta M = 15.003\,065 - 15.000\,109 = 0.002\,956\,u \qquad \Delta m = 1.008\,665 - 1.007\,825 = 0.000\,840\,u$$

$$\Delta M + \Delta m = 0.003\,796(1.660\,54 \times 10^{-27}) = 6.30 \times 10^{-30}\,kg$$

$$\begin{aligned}R &= \frac{3(7)(1.602 \times 10^{-19})^2}{10\pi(8.854 \times 10^{-12})(6.30 \times 10^{-30})(3 \times 10^8)^2}\\ &= \frac{5.389 \times 10^{-37}}{1.578 \times 10^{-22}}\\ &= 3.415 \times 10^{-15}\,m\\ &= 3.415\,fm\end{aligned}$$

The radii of the mirror isobars are $3.42\,fm$.