Question

Photons and Gravity: When a nucleus emits a photon, it recoils in the opposite direction. (a) The $^{57}_{27}$Co nucleus decays by K capture to $^{57}_{26}$Fe, which then emits a photon in losing 14.4 keV to reach its ground state. The mass of a $^{57}_{26}$Fe atom is $9.5 \times 10^{-26}\,kg$. By how much is the photon energy reduced from the full $14.4\,keV$ available as a result of having to share energy and momentum with the recoiling atom? (b) In certain crystals the atoms are so tightly bound that the entire crystal recoils when a gamma-ray photon is emitted, instead of the individual atom. This phenomenon is known as the *Mössbauer effect*. By how much is the photon energy reduced in this situation if the excited $^{57}_{26}$Fe nucleus is part of a 1.0 g crystal? (c) An experiment measures the increase in frequency of photons 'falling' in a gravitational field. What is the original frequency and change in frequency of a $14.4\,keV$ gamma-ray photon after it has fallen 20 m near to the earth's surface?

Solution

a) The momenta of the $^{57}_{26}$Fe nucleus and the γ-photon are equal and opposite, hence

$$p = h\nu/c \quad \text{or} \quad pc = h\nu \qquad (1)$$

Where $h\nu$ is the *actual* energy of the emitted photon.

The total energy released $E(14.4\,keV)$ is the KE of the $^{57}_{26}$Fe nucleus plus the energy of the γ-photon, hence

$$\begin{aligned}
E &= \left(\sqrt{m^2c^4 + p^2c^c} - mc^2\right) + h\nu \\
&= \left(\sqrt{m^2c^4 + h^2\nu^2} - mc^2\right) + h\nu \qquad \text{subs from (1)} \\
(E + mc^2 - h\nu)^2 &= m^2c^4 + h^2\nu^2 \\
2mc^2h\nu + 2Eh\nu &= 2Emc^2 + E^2 \\
h\nu &= E\left(\frac{2mc^2 + E}{2mc^2 + 2E}\right) \\
&= E\left(1 + \frac{E}{2mc^2}\right)\left(1 + \frac{E}{mc^2}\right)^{-1} \\
&= E\left(1 + \frac{E}{2mc^2}\right)\left(1 - \frac{E}{mc^2} + \left(\frac{E}{mc^2}\right)^2 + ...\right) \\
&\approx E\left(1 - \frac{E}{mc^2} + \frac{E}{2mc^2}\right) \\
h\nu &= E\left(1 - \frac{E}{2mc^2}\right)
\end{aligned}$$

Hence the energy lost to KE is $E^2/2mc^2$.

The mass of of $^{57}_{26}$Fe nucleus is $9.5 \times 10^{-26}kg$ ($9.5 \times 10^{-26}c^2 = 8.55 \times 10^{-9}J$ or $5.34 \times 10^{10}\,eV$).

$$\text{Energy lost} = \frac{E^2}{2mc^2} = \frac{(14.4 \times 10^3)^2}{2(5.34 \times 10^{10})} = 1.94 \times 10^{-3}$$

The photon energy is reduced by $1.9 \times 10^{-3}\,eV$

b) A $1g$ crystal has mass of $Mc^2 = 9 \times 10^{13}\, J$ or $5.62 \times 10^{32}\, eV$

$$\begin{aligned} \text{Energy lost} &= \frac{E^2}{2Mc^2} \\ &= \frac{(14.4 \times 10^3)^2}{2(5.62 \times 10^{32})} \\ &= 1.84 \times 10^{-25}\, eV \end{aligned}$$

The energy of the photon is reduced by only $1.8 \times 10^{-25}\, eV$

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c) A photon of of energy $14.4\, keV$ has an initial frequency of $1.44 \times 10^4/h = 1.44 \times 10^4/4.136 \times 10^{-15}$ or $3.48 \times 10^{18}\, Hz$. The photon energy after falling through a height H is given by (see §2.9 in the text)

$$\begin{aligned} h\nu' &= h\nu \left(1 + \frac{gH}{c^2}\right) \\ \nu' - \nu &= h\nu \frac{gH}{c^2} \\ &= 3.48 \times 10^{18} \left(\frac{9.81 \times 20}{(3 \times 10^8)^2}\right) \\ &= 7.59 \times 10^3 \end{aligned}$$

The original frequency was $3.5 \times 10^{18}\, Hz$ and the change in frequency $7.6\, kHz$
