Question

Uncertainty Principle: The frequency of oscillation of a harmonic oscillator of mass m and spring constant C is $\nu = \sqrt{C/m}/2\pi$. The energy of the oscillator is $E = p^2/2m + Cx^2/2$, where p is its momentum when its displacement from the equilibrium position is x. In classical physics the minimum energy of the oscillator is $E_{min} = 0$. Use the uncertainty principle to find an expression for E in terms of x only and show that the minimum energy is actually $E_{min} = h\nu/2$ by setting $dE/dx = 0$ and solving for E_{min}.

Solution

For minimum displacement and momentum the uncertainty principle $\Delta x \Delta p \geq \hbar/2$ may be taken as $xp \approx \hbar/2$, hence

$$\begin{aligned} E &= p^2/2m + Cx^2/2 \\ &= \frac{1}{2m}\left(\frac{\hbar}{2x}\right)^2 + \frac{Cx^2}{2} \\ &= \frac{\hbar^2}{8mx^2} + \frac{Cx^2}{2} \end{aligned}$$

$$\begin{aligned} \frac{dE}{dx} &= \frac{\hbar^2}{8m}\left(\frac{-2}{x^3}\right) + Cx \\ &= -\frac{\hbar^2}{4mx^3} + Cx \end{aligned}$$

For minimum energy $dE/dx = 0$, hence (using $\nu = \sqrt{C/m}/2\pi$)

$$\begin{aligned} Cx &= \frac{\hbar^2}{4mx^3} \\ x^4 &= \frac{\hbar^2}{4mC} \\ &= \frac{\hbar^2}{4m}\left(\frac{1}{4\pi^2\nu^2 m}\right) \\ &= \frac{\hbar^2}{16\pi^2 m^2 \nu^2} \\ x^2 &= \frac{\hbar}{4\pi m\nu} \end{aligned}$$

Subs for x^2 and C in E

$$\begin{aligned} E_{min} &= \frac{\hbar^2}{8mx^2} + \frac{Cx^2}{2} \\ &= \frac{\hbar^2}{8m}\left(\frac{4\pi m\nu}{\hbar}\right) + \frac{4\pi^2\nu^2 m}{2}\left(\frac{\hbar}{4\pi m\nu}\right) \\ &= \frac{\pi\hbar\nu}{2} + \frac{\pi\hbar\nu}{2} \\ &= \pi\hbar\nu \quad \text{or} \quad h\nu/2 \end{aligned}$$

The minimum energy for a harmonic oscillator is $h\nu/2$.