Question

The Bohr Atom: The fine structure constant is defined as $\alpha = e^2/2\epsilon_0 hc$. (a) Show that $\alpha = v_1/c$ where v_1 is the velocity of the electron in the ground state of the Bohr atom. (b) Show that the value of α is very close to 1/137 and is a pure number with no dimensions. (c) Show that $\alpha a_0 = \lambda_c/2\pi$, where a_0 is the radius of the ground-state Bohr orbit and λ_C is the Compton wavelength of the electron.

Solution

Bohr atom

$$a_0 = r_1 = \frac{h^2\epsilon_0}{\pi m e^2} = 5.292 \times 10^{-11}\, m$$

$$v_1 = \frac{e}{\sqrt{4\pi\epsilon_0 m r_1}} = \frac{1.602 \times 10^{-19}}{\sqrt{4\pi(8.854 \times 10^{-12})(9.1095 \times 10^{-31})(5.292 \times 10^{-11})}} = 2.187 \times 10^6\, m/s$$

a) & b)

$$\begin{aligned} v_1 &= \frac{e}{\sqrt{4\pi\epsilon_0 m r_1}} \\ &= e\sqrt{\frac{1}{4\pi\epsilon_0 m}\,\frac{\pi m e^2}{h^2\epsilon_0}} \\ &= \frac{e^2}{2\epsilon_0 h} \\ \frac{v_1}{c} &= \frac{e^2}{2\epsilon_0 hc} \\ \rightarrow \quad \alpha &= \frac{v_1}{c} \end{aligned}$$

$$\begin{aligned} \alpha &= \frac{2.187 \times 10^6}{3 \times 10^8} \\ &\approx 1/137 \end{aligned}$$

The fine structure constant is close to 1/137 and is clearly dimensionless.

c) Compton wavelength for electron $\lambda_C = h/mc$

$$\begin{aligned} \alpha a_0 &= \frac{e^2}{2\epsilon_0 hc}\,\frac{h^2\epsilon_0}{\pi m e^2} \\ &= \frac{h}{2\pi mc} \\ &= \frac{\lambda_C}{2\pi} \end{aligned}$$