Question

Harmonic Oscillator: Find the expectation values $\langle x\rangle$ and $\langle x^2\rangle$ for the first two states of a harmonic oscillator.

Solution

For $n = 0$

$$\psi_0 = \left(\frac{2m\nu}{\hbar}\right)^{1/4} e^{-(\pi m\nu/\hbar)x^2} \qquad E_0 = \frac{h\nu}{2} \quad \left(\nu = \frac{1}{2\pi}\sqrt{\frac{m}{k}}\right) \qquad *$$

Hence the expectation values

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x|\psi_0|^2 dx \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/2} \int_{-\infty}^{\infty} xe^{-(2\pi m\nu/\hbar)x^2} dx \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/2} \left(\frac{\hbar}{2\pi m\nu}\right) \int_{-\infty}^{\infty} ye^{-y^2} dy \qquad \text{where } y = \left(\frac{2\pi m\nu}{\hbar}\right)^{1/2} x \\
&= 0 \qquad **
\end{aligned}$$

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} x^2|\psi_0|^2 dx \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/2} \left(\frac{\hbar}{2\pi m\nu}\right)^{3/2} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy \\
&= \frac{\hbar}{2\pi^{3/2} m\nu} \frac{\sqrt{\pi}}{2} \qquad ** \\
&= \frac{\hbar}{4\pi m\nu} \qquad \text{then subs for } m \\
&= \frac{\hbar}{4\pi\nu} \times \frac{4\pi^2\nu^2}{k} \\
&= \frac{h\nu}{2k} \quad \text{or} \quad \frac{E_0}{k}
\end{aligned}$$

For $n = 1$

$$\psi_1 = 2\left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{\pi m\nu}{\hbar}\right)^{1/2} xe^{-(\pi m\nu/\hbar)x^2} \qquad E_1 = \frac{3h\nu}{2} \quad \left(\nu = \frac{1}{2\pi}\sqrt{\frac{m}{k}}\right) \qquad *$$

Hence the expectation values

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} x|\psi_1|^2 dx \\
&= 4\left(\frac{2m\nu}{\hbar}\right)^{1/2} \left(\frac{\pi m\nu}{\hbar}\right) \int_{-\infty}^{\infty} x^3 e^{-2(\pi m\nu/\hbar)x^2} dx \\
&= 4\left(\frac{2m\nu}{\hbar}\right)^{1/2} \left(\frac{\pi m\nu}{\hbar}\right) \left(\frac{\hbar}{2\pi m\nu}\right)^2 \int_{-\infty}^{\infty} y^3 e^{-y^2} dy \\
&= 0 \qquad **
\end{aligned}$$

$$
\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\psi_1|^2 dx \\
&= 4 \left(\frac{2m\nu}{\hbar} \right)^{1/2} \left(\frac{\pi m \nu}{\hbar} \right) \left(\frac{\hbar}{2\pi m \nu} \right)^{5/2} \int_{-\infty}^{\infty} y^4 e^{-y^2} dy \\
&= \frac{\hbar}{\pi^{3/2} m \nu} \frac{3\sqrt{\pi}}{4} \qquad ** \\
&= \frac{3\hbar}{4\pi m \nu} \qquad \text{then subs for } m \\
&= \frac{3\hbar}{4\pi\nu} \times \frac{4\pi^2 \nu^2}{k} \\
&= \frac{3h\nu}{2k} \quad \text{or} \quad \frac{E_1}{k}
\end{aligned}
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* These have been calculated from Eq 5.72 in the text using Table 5.2, ν is the classical frequency for a harmonic oscillator with restoring force k proportional to displacement x.

** $\int y^n e^{-y^2} dy$ is always zero for n odd and is easily calculated (or looked up) for n even.