Question

Harmonic Oscillator: Schrödinger's equation for a harmonic oscillator is, with $U = \frac{1}{2}kx^2$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi = 0$$

Verify the following solution for $n = 2$

$$\psi_n = \left(\frac{2m\nu}{\hbar}\right)^{1/4} (2^n n!)^{-1/2}\ H_n(y)e^{-y^2/2}$$

Where

$$y = \left(\frac{1}{\hbar}\sqrt{km}\right)^{1/2} x \qquad \text{and} \qquad H_2(y) = 4y^2 - 2 \quad \text{(Hermite polynomial)}$$

Solution

Using the above substitution for y together with $\alpha_n = \dfrac{2E_n}{\hbar}\sqrt{\dfrac{m}{k}} = \dfrac{2E_n}{h\nu}$ where ν is the classical frequency of the oscillator, Schrödinger's equation reduces to (see §5.11 in text)

$$\frac{d^2\psi_n}{dy^2} + (\alpha_n - y^2)\psi_n = 0 \qquad \text{where} \qquad \alpha_n = 2n + 1\ (n = 0, 1, 2, 3\,...)$$

Subs $n = 2$ in the solution

$$\begin{aligned}
\psi_2 &= \left(\frac{2m\nu}{\hbar}\right)^{1/4} (2^2\,2!)^{-1/2}(4y^2 - 2)e^{-y^2/2} \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{1}{8}\right)^{1/2} (4y^2 - 2)e^{-y^2/2} \\
\frac{d\psi_2}{dy} &= \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{1}{8}\right)^{1/2} \left(4y^2e^{-y^2/2}(-y) + 8ye^{-y^2/2} - 2e^{-y^2/2}(-y)\right) \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{1}{8}\right)^{1/2} (-4y^3 + 10y)e^{-y^2/2} \\
\frac{d^2\psi_2}{dy^2} &= \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{1}{8}\right)^{1/2} \left(-4y^3e^{-y^2/2}(-y) - 12y^2e^{-y^2/2} + 10ye^{-y^2/2}(-y) + 10e^{-y^2/2}\right) \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{1}{8}\right)^{1/2} (4y^4 - 22y^2 + 10)e^{-y^2/2} \\
&= \left(\frac{2m\nu}{\hbar}\right)^{1/4} \left(\frac{1}{8}\right)^{1/2} (y^2 - 5)(4y^2 - 2)e^{-y^2/2} \\
&= -(5 - y^2)\psi_2 \\
\frac{d^2\psi_2}{dy^2} &= -(\alpha_2 - y^2)\psi_2
\end{aligned}$$

Hence the solution is valid for $n = 2$.