Question

Quantum Mechanics: The wave function of a certain particle is $\psi = A\,cos^2 x$ for $-\pi/2 < x < \pi/2$. (a) Find the value of A. (b) Find the probability that the particle be found between $x = 0$ and $x = \pi/4$.

Solution

a) Probability of finding the particle somewhere in the region

$$\begin{aligned}
P_{x_1,x_2} &= \int_{-\pi/2}^{\pi/2} |\psi|^2\,dx \\
&= A^2 \int_{-\pi/2}^{\pi/2} cos^4 x\,dx \\
&= \frac{A^2}{4} \int_{-\pi/2}^{\pi/2} (1 + cos2x)^2\,dx \\
&= \frac{A^2}{4} \int_{-\pi/2}^{\pi/2} (1 + 2cos2x + cos^2 2x)\,dx \\
&= \frac{A^2}{4} \int_{-\pi/2}^{\pi/2} \Big(1 + 2cos2x + \tfrac{1}{2}(1 + cos4x)\Big)\,dx \\
&= \frac{A^2}{4} \left[x + sin2x + \frac{x}{2} + \frac{sin4x}{8}\right]_{-\pi/2}^{\pi/2} \\
&= \frac{A^2}{4} \left[\frac{3\pi}{4} + \frac{3\pi}{4}\right] \\
&= \frac{3\pi A^2}{8}
\end{aligned}$$

For normalization this integral must be unity i.e. $A = \sqrt{\dfrac{8}{3\pi}}$

b) Probability of finding the particle between 0 and $\pi/4$

$$\begin{aligned}
P_{x_1,x_2} &= \int_{0}^{\pi/4} |\psi|^2\,dx \\
&= \frac{8}{3\pi} \int_{0}^{\pi/4} cos^4 x\,dx \\
&= \frac{8}{3\pi}\,\frac{1}{4} \left[x + sin2x + \frac{x}{2} + \frac{sin4x}{8}\right]_{0}^{\pi/4} \\
&= \frac{2}{3\pi} \left[\frac{3\pi}{8} + 1\right] \\
&= \frac{1}{4} + \frac{2}{3\pi} \\
&= 0.462
\end{aligned}$$

The probability of finding the particle in this region is 0.46