Question

Quantum Numbers: The wave functions of the hydrogen atom are given by $\psi = R_{nl}\Theta_{lm_l}\Phi_{m_l}$ where n, l, m_l are the principal, orbital and magnetic quantum numbers respectively. Show that

$$R_{10}(r) = \frac{2}{a_0^{3/2}}\, e^{-r/a_0}$$

is a solution to Schrödinger's equation for R

$$\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) - \frac{l(l+1)}{r^2}\right] R = 0$$

and that it is normalized.

Solution

For $n = 1$, $l = 0$

$$E = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2} = -\frac{\hbar^2}{2ma_0^2} \qquad \text{where } a_0 \text{ (Bohr radius)} = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

$$\frac{dR}{dr} = -\frac{2}{a_0^{5/2}}e^{-r/a_0} \qquad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{2}{a_0^{5/2}}\left(\frac{-r^2}{a_0} + 2r\right)e^{-r/a_0}$$

$$\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) - \frac{l(l+1)}{r^2} = \frac{2me^2}{4\pi\epsilon_0\hbar^2 r} + \frac{2mE}{\hbar^2} = \frac{2}{a_0 r} - \frac{1}{a_0^2}$$

Subs these values in Schrödinger's equation

$$\frac{1}{r^2}\left[-\frac{2}{a_0^{5/2}}\left(\frac{-r^2}{a_0} + 2r\right)\right]e^{-r/a_0} + \left[\frac{2}{a_0 r} - \frac{1}{a_0^2}\right]\frac{2}{a_0^{3/2}}\,e^{-r/a_0}$$

$$= \left[\frac{2}{a_0^{7/2}} - \frac{4}{r a_0^{5/2}}\right]e^{-r/a_0} + \left[\frac{4}{r a_0^{5/2}} - \frac{2}{a_0^{7/2}}\right]e^{-r/a_0} \quad = 0$$

Hence R_{10} satisfies Schrödinger's equation for R.

The probability of finding an electron in a spherical shell between r and $(r + dr)$ is $r^2|R|^2dr$

$$\begin{aligned}
\int_0^\infty r^2|R|^2dr &= \frac{4}{a_0^3}\int_0^\infty r^2 e^{-2r/a_0} \\
&= \frac{4}{a_0^3}\left[\frac{-a_0 r^2}{2}e^{-2r/a_0}\right]_0^\infty - \frac{4}{a_0^3}\int_0^\infty -a_0 r e^{-2r/a_0}dr \\
&= \frac{4}{a_0^2}\int_0^\infty r e^{-2r/a_0}dr \\
&= \frac{4}{a_0^2}\left[\frac{-a_0 r}{2}e^{-2r/a_0}\right]_0^\infty - \frac{4}{a_0^2}\int_0^\infty -\frac{a_0}{2}e^{-2r/a_0} \\
&= \frac{2}{a_0}\int_0^\infty e^{-2r/a_0}dr \\
&= -\left[e^{-2r/a_0}\right]_0^\infty = -(0-1) = 1
\end{aligned}$$

Hence R_{10} is also normalized.