Question

Total Angular Momentum: The magnetic moment $\boldsymbol{\mu}_J$ of an atom in which LS coupling holds has the magnitude

$$\mu_J = \sqrt{\mathsf{J(J+1)}}\, g_J \mu_B$$

where $\mu_B = e\hbar/2m$ is the Bohr magneton and

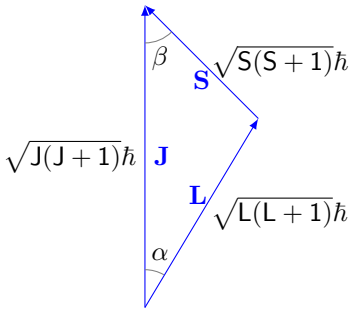
$$g_J = 1 + \frac{\mathsf{J(J+1) - L(L+1) + S(S+1)}}{\mathsf{2J(J+1)}}$$

is the **Landé g factor**. (a) Derive this result with the help of the law of cosines starting from the fact that averaged over time, only the components of $\boldsymbol{\mu}_L$ and $\boldsymbol{\mu}_S$ parallel to $\mathbf{J}$ contribute to $\boldsymbol{\mu}_J$. (b) Consider an atom that obeys LS coupling that is in a weak magnetic field $\mathbf{B}$ in which coupling is preserved. How many substates are there for a given value of J? What is the energy difference between substates?

Solution

a) Angular magnetic moment

$$\begin{aligned}\boldsymbol{\mu}_L &= -\frac{e}{2m}\mathbf{L} \\ \mu_L &= \frac{e\hbar}{2m}\sqrt{\mathsf{L(L+1)}} \\ &= \mu_B \sqrt{\mathsf{L(L+1)}}\end{aligned}$$



Spin magnetic moment

$$\begin{aligned}\boldsymbol{\mu}_S &= -\frac{e}{m}\mathbf{S} \\ \mu_S &= \frac{e\hbar}{m}\sqrt{\mathsf{S(S+1)}} \\ &= 2\mu_B \sqrt{\mathsf{S(S+1)}}\end{aligned}$$

Law of cosines

$$cos\alpha = \frac{\mathsf{J(J+1) + L(L+1) - S(S+1)}}{\mathsf{2\sqrt{J(J+1)L(L+1)}}} \qquad cos\beta = \frac{\mathsf{J(J+1) + S(S+1) - L(L+1)}}{\mathsf{2\sqrt{J(J+1)S(S+1)}}}$$

Averaging over time i.e. taking components of $\mathbf{S}$ and $\mathbf{L}$ in the direction of $\mathbf{J}$

$$\begin{aligned}\mu_J &= \mu_L\, cos\alpha + \mu_S\, cos\beta \\ &= \mu_B \sqrt{\mathsf{L(L+1)}}\ \frac{\mathsf{J(J+1) + L(L+1) - S(S+1)}}{\mathsf{2\sqrt{J(J+1)L(L+1)}}} + 2\mu_B \sqrt{\mathsf{S(S+1)}}\ \frac{\mathsf{J(J+1) + S(S+1) - L(L+1)}}{\mathsf{2\sqrt{J(J+1)S(S+1)}}} \\ &= \frac{\mu_B}{\mathsf{2\sqrt{J(J+1)}}}\Big(\mathsf{3J(J+1) - L(L+1) + S(S+1)}\Big) \\ &= \sqrt{\mathsf{J(J+1)}}\left(\frac{\mathsf{2J(J+1) + J(J+1) - L(L+1) + S(S+1)}}{\mathsf{2J(J+1)}}\right)\mu_B \\ &= \sqrt{\mathsf{J(J+1)}}\left(1 + \frac{\mathsf{J(J+1) - L(L+1) + S(S+1)}}{\mathsf{2J(J+1)}}\right)\mu_B \\ &= \sqrt{\mathsf{J(J+1)}}\, g_J\, \mu_B \qquad \text{where } g_J \text{ is the Landé g factor.}\end{aligned}$$

b) For LS coupling

$$J = \sqrt{\mathsf{J(J+1)}}\,\hbar \qquad\qquad J_Z = \mathsf{M}_J\,\hbar$$

Where $M_J = -J, -J+1, ..., J-1, J$

J is half integral if an odd number of electrons are involved; integral (or 0) if an even number of electrons are involved. Either way there are $2\mathsf{J}+1$ substates in the presence of a magnetic field, $\mathbf{B}$.

The magnetic energy of substate $U_{M_J} = \mu_J B cos\theta$, where θ is the angle between $\boldsymbol{\mu}_J$ and $\mathbf{B}$. Since space is quantized the permitted values of θ are given by by $cos\theta = \mathsf{M}_J/\sqrt{\mathsf{J(J+1)}}$, hence

$$\begin{aligned} U_{M_J} &= \mu_J B cos\theta \\ &= \sqrt{\mathsf{J(J+1)}}\, g_J\, \mu_B\, B\, \frac{\mathsf{M}_J}{\sqrt{\mathsf{J(J+1)}}} \\ &= g_J\, \mu_B\, B\, \mathsf{M}_J \qquad * \end{aligned}$$

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* This is answer given in the text for ΔE, the energy difference between substates, if this means the difference between two successive substates i.e. $\Delta M_J \pm 1$, then $\Delta E = g_J\, \mu_B\, B$.