Question

Molecular Energies in an Ideal Gas: Verify that the most probable speed of an ideal gas molecule is $\sqrt{2kT/m}$.

Solution

In an ideal gas containing N molecules (Maxwell-Boltzmann distribution) the number of molecules energies ϵ to $\epsilon + d\epsilon$, absolute temperature T is

$$n(\epsilon)d\epsilon = \frac{2\pi N}{(\pi kT)^{3/2}} \sqrt{\epsilon}\, e^{-\epsilon/kT} d\epsilon$$

Since $\epsilon = \frac{1}{2}mv^2 \qquad d\epsilon = mv\, dv$ the number with a given speed v to $v + dv$ is

$$n(v)dv = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv$$

The most probable speed v_p is given by setting $\dfrac{d}{dv} n(v) = 0$ hence

$$\begin{aligned} 2ve^{-mv^2/2kT} + v^2 e^{-mv^2/2kT}(-mv/kT) &= 0 \\ 2v - \frac{mv^3}{kT} &= 0 \\ v &= \sqrt{\frac{2kT}{m}} \end{aligned}$$

The most probable speed of an ideal gas molecule $v_p = \sqrt{2kT/m}$.