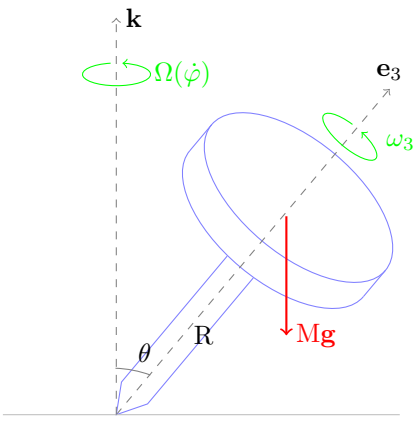
Problem

Find the Lagrangian function for a symmetric top whose pivot is free to slide on a smooth horizontal table, in terms of the generalized co-ordinates $X, Y, \varphi, \theta, \psi$, and the principal moments I_1^*, I_2^*, I_3^* about the centre of mass. (Note that Z is related to θ.) Show that the horizontal motion of the centre of mass may be completely separated from the rotational motion. What difference is there in the equation for the steady precession of a top with fixed pivot ($I_1\Omega^2 cos\theta - I_3\omega_3\Omega + MgR = 0$)? Are the precessional angular velocities greater or less than in the case of a fixed pivot? Show that steady precession at a given value of θ can occur for a smaller value of ω_3 than in the case of a fixed pivot.

Solution

The motion of the CM of the top is given by co-ordinates (X,Y,Z) and its orientation in space specified by the Euler angles (φ, θ, ψ). Hence the Lagrangian



$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}M\dot{Y}^2 + \tfrac{1}{2}M\dot{Z}^2 + \tfrac{1}{2}I_1^*\omega_1^2 + \tfrac{1}{2}I_2^*\omega_2^2 + \tfrac{1}{2}I_3^*\omega_3^2 - MgZ$$

Where

$$Z = Rcos\theta \qquad \dot{Z} = -Rsin\theta\,\dot{\theta}$$

$$\boldsymbol{\omega} = -\dot{\varphi}sin\theta\mathbf{e'}_1 + \dot{\theta}\mathbf{e'}_2 + (\dot{\psi} + \dot{\varphi}cos\theta)\mathbf{e}_3$$

Hence

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}M\dot{Y}^2 + \tfrac{1}{2}MR^2sin^2\theta\,\dot{\theta}^2 + \tfrac{1}{2}I_1^*(\dot{\varphi}sin\theta)^2 + \tfrac{1}{2}I_1^*\dot{\theta}^2 + \tfrac{1}{2}I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)^2 - MgRcos\theta$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X} \quad \rightarrow \quad M\ddot{X} = 0 \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) = \frac{\partial L}{\partial Y} \quad \rightarrow \quad M\ddot{Y} = 0 \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left(MR^2sin^2\theta\,\dot{\theta} + I_1^*\,\dot{\theta}\right) = MR^2sin\theta cos\theta\,\dot{\theta}^2 + I_1^*sin\theta cos\theta\dot{\varphi}^2 - I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)\dot{\varphi}sin\theta + MgRsin\theta$$

$$(MR^2sin^2\theta + I_1^*)\ddot{\theta} + MR^2sin\theta cos\theta\,\dot{\theta}^2 = I_1^*sin\theta cos\theta\dot{\varphi}^2 - I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)\dot{\varphi}sin\theta + MgRsin\theta \tag{3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}$$

$$\frac{d}{dt}\left(I_1^*\dot{\varphi}sin^2\theta + I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)cos\theta\right) = 0 \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{\partial L}{\partial \psi}$$

$$\frac{d}{dt}\left(I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)\right) = 0 \tag{5}$$

Equations (1) & (2) specifying the horizontal motion of the CM are independent of the rotational motion specified by (3), (4) & (5); and vice versa, hence the two motions may be completely separated.

The right hand side of (3) is identical to that for the steady precession about a fixed pivot with the moments of inertia about the pivot being replaced by those about the CM. ($I_1^* < I_1$ and $I_3^* = I_3$)

Equation (5) shows $\omega_3\,(\dot{\psi} + \dot{\varphi} cos\theta)$ is constant. For steady precession (θ constant) equations (4) & (5) then indicate $\dot{\varphi}(\Omega)$ is also constant.

For steady precession the left hand side of (3) must vanish, hence (for $sin\theta \neq 0$)

$$I_1^* cos\theta\,\Omega^2 - I_3^*\omega_3\,\Omega + MgR = 0$$

and the precessional angular velocities

$$\Omega = \frac{I_3^*\omega_3 \pm I_3^*\omega_3\big(1 - 4I_1^* MgR cos\theta/(I_3^*\omega_3)^2\big)^{1/2}}{2I_1^* cos\theta}$$

$$\Omega_+ = \frac{I_3^*\omega_3}{I_1^* cos\theta} - \frac{mgR}{I_3^*\omega_3} + \mathcal{O}(\omega_3^{-3}) \qquad \Omega_- = \frac{mgR}{I_3^*\omega_3} + \frac{I_1^* cos\theta (MgR)^2}{(I_3^*\omega_3)^3} + \mathcal{O}(\omega_3^{-5})$$

Since $I_1^* < I_1$ and $I_3^* = I_3$ the dominant first term for the rapid precession is larger i.e. Ω_+ is higher .

For the slower precession Ω_- is unchanged for large values of ω_3, but the second term shows that in general it is slightly smaller i.e. Ω_- is slightly lower.

The minimum value for real Ω is given by

$$(I_3^*\omega_3)^2 = 4I_1^* MgR cos\theta$$
$$\omega_3 = \frac{2\sqrt{I_1^* mgR cos\theta}}{I_3^*}$$

Hence again, since $I_1^* < I_1$ and $I_3^* = I_3$, ω_3 is slightly smaller i.e. for a given θ steady precession can occur for a lower value of ω_3 than for the fixed pivot.
