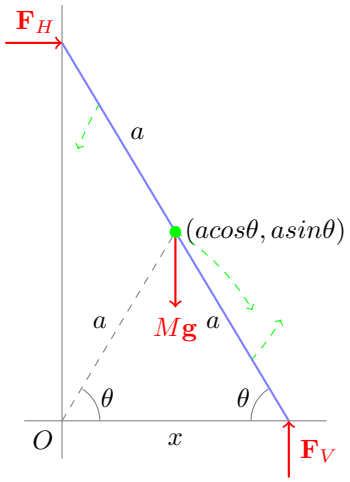
Problem

A uniform plank of length $2a$ is placed with one end on a smooth horizontal floor and the other against a smooth vertical wall. Write down the Lagrangian function, using two generalized co-ordinates, the distance x of the foot of the plank from the wall, and its angle θ of inclination to the horizontal, with a suitable constraint between the two. Given that the plank is initially at rest at an inclination of $60°$, find the angle at which it loses contact with the wall. (*Hint*: First write the co-ordinates of the centre of mass in terms of x and θ. Note that the reaction at the wall is related to the Lagrange multiplier.)

Solution

The plank rotates about its CM (X,Y) velocity $\dot\theta$ and the trajectory of the CM is a circle radius a about O, for as long as it stays in contact with the wall. Hence



$$L = \tfrac{1}{2}M\dot X^2 + \tfrac{1}{2}M\dot Y^2 + \tfrac{1}{2}I_1^*\dot\theta^2 - Mgasin\theta \qquad *$$
$$= \tfrac{1}{2}M(-asin\theta\,\dot\theta)^2 + \tfrac{1}{2}M(acos\theta\,\dot\theta)^2 + \tfrac{1}{2}\left(\frac{Ma^2}{3}\right)\dot\theta^2 - Mgasin\theta$$
$$= \tfrac{2}{3}Ma^2\,\dot\theta^2 - Mgasin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{\partial L}{\partial\theta}$$
$$\tfrac{4}{3}Ma^2\ddot\theta = -Mgacos\theta$$
$$\ddot\theta = -\frac{3gcos\theta}{4a} \qquad (1)$$
$$\ddot\theta\,\dot\theta = -\frac{3gcos\theta\,\dot\theta}{4a}$$
$$\dot\theta^2 = -\frac{3gsin\theta}{2a} + C \qquad t=0,\ \theta = 60°,\ \dot\theta = 0 \quad\rightarrow\quad C = 3gsin\,60°/2a$$
$$\dot\theta^2 = \frac{3g}{2a}\left(\frac{\sqrt3}{2} - sin\theta\right) \qquad (2)$$

When the plank loses contact $\mathbf{F}_H \rightarrow 0$ and it ceases to accelerate horizontally i.e. $\ddot x = 0$

$$x = 2acos\theta$$
$$\dot x = -2asin\theta\,\dot\theta$$
$$\ddot x = -2a(sin\theta\,\ddot\theta + cos\theta\,\dot\theta^2)$$

Hence the plank loses contact when $sin\theta\,\ddot\theta = -cos\theta\,\dot\theta^2$. Subs from (1) & (2) gives

$$sin\theta\,\frac{3gcos\theta}{4a} = cos\theta\,\frac{3g}{2a}\left(\frac{\sqrt3}{2} - sin\theta\right)$$
$$sin\theta = \frac{\sqrt3}{3} = \frac{1}{\sqrt3}$$
$$\theta = 35.26°$$

The plank loses contact with the wall when its inclination to the horizontal is $35.3°$.

* Using the method suggested in the text, with λ as Lagrange multiplier

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}M\dot{Y}^2 + \tfrac{1}{2}I_1^*\dot{\theta}^2 - MgY - \lambda(X - acos\theta)$$

Where $X = x/2$ Y=$asin\theta$ $\dot{X} = \dot{x}/2$ $\dot{Y} = acos\theta\,\dot{\theta}$

$$L = \frac{M}{8}\dot{x}^2 + \frac{M}{2}a^2cos^2\theta\,\dot{\theta}^2 + \frac{Ma^2}{6}\dot{\theta}^2 - Mgasin\theta - \lambda(\frac{x}{2} - acos\theta)$$

or similar.

This expression contains x, θ, and their derivatives, so Hamilton's principle may be used. (The *Hint* is a bit confusing in this problem!)
