Problem

Two bodies of masses M_1 and M_2 are moving in circular orbits of radii a_1 and a_2 about their centre of mass, a third small body of mass m ($<< M_1$ or M_2) is moving in the gravitational field of the first two and in the same plane. Find the equations that must be satisfied for 'equilibrium' in the rotating frame (i.e., circular motion with the same angular velocity as M_1 and M_2). Consider 'equilibrium' positions on the line of centres of M_1 and M_2. By roughly sketching the effective potential energy curve, show that there are three such positions, but that all three are unstable. (Note: The positions are actually the solutions of a quintic equation.) Show also that there are two 'equilibrium' positions off the line of centres, in each of which the three bodies form an equilateral triangle. (Lagrangian points.)

Solution

The equations of motion for m (from Problem 15) are

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = -\frac{GM_1}{r_1^3}(x + a_1) - \frac{GM_2}{r_2^3}(x - a_2) \tag{1}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{GM_1}{r_1^3}y - \frac{GM_2}{r_2^3}y \tag{2}$$

Where $\quad r_1 = \left((x + a_1)^2 + y^2\right)^{1/2} \quad r_2 = \left((x - a_2)^2 + y^2\right)^{1/2} \quad \omega^2 = GM/a^3 \quad a = a_1 + a_2$

Taking the velocity of m as zero for 'equilibrium' positions and re-arranging

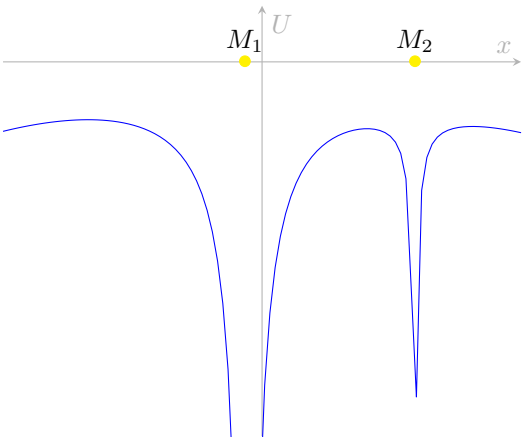
$$\ddot{x} = -\frac{GM_1}{r_1^3}(x + a_1) - \frac{GM_2}{r_2^3}(x - a_2) + \omega^2 x \tag{3}$$

$$\ddot{y} = -\frac{GM_1}{r_1^3}y - \frac{GM_2}{r_2^3}y + \omega^2 y \tag{4}$$

The right hand side of these equations is equivalent to the negative gradient of an effective potential $U = V_g + V_c$, where V_g is the normal gravitational potential and V_c the contribution from the centrifugal term. Hence along the line of centres (x axis)

$$U = -\frac{GM_1}{x + a_1} - \frac{GM_2}{x - a_2} - \tfrac{1}{2}\omega^2 x^2$$

A typical plot of the effective potential energy with $M_1 = 10M_2$ shows the positions of the three maxima.



These are the 'equilibrium' points on the extended line of centres between the two masses.

The shape of the curve indicates that all three are unstable. If a small mass at any of these positions is disturbed it will not return to its original position.

For all five 'equilibrium' positions the force acting on m must be zero, hence from (3) & (4)

$$-\frac{GM_1}{r_1^3}(x+a_1)-\frac{GM_2}{r_2^3}(x-a_2)+\omega^2 x=0 \qquad (5a)$$

$$-\frac{GM_1}{r_1^3}y-\frac{GM_2}{r_2^3}y+\omega^2 y=0 \qquad (6a)$$

Using the identities $GM_1=\omega^2a^2a_2$ and $GM_2=\omega^2a^2a_1$ (Problem 25)

$$\frac{a^2a_2(x+a_1)}{r_1^3}+\frac{a^2a_1(x-a_2)}{r_2^3}-x=0 \qquad (5b)$$

$$\frac{a^2a_2}{r_1^3}+\frac{a^2a_1}{r_2^3}-1=0 \qquad (6b)$$

Three of the solutions are already known from above to be on the x-axis, a sensible place to look for solutions off the line of centres is on the line equidistant from the two masses i.e. $x=\frac{1}{2}(a_2-a_1)$, where $r_1=r_2$, and thus simplify the equations. Subs for x gives

$$r_1=\left(\tfrac{1}{4}(a_1+a_2)^2+y^2\right)^{1/2}=\left(\tfrac{1}{4}(-a_1-a_2)^2+y^2\right)^{1/2}=r_2=\left(\tfrac{1}{4}a^2+y^2\right)^{1/2}$$

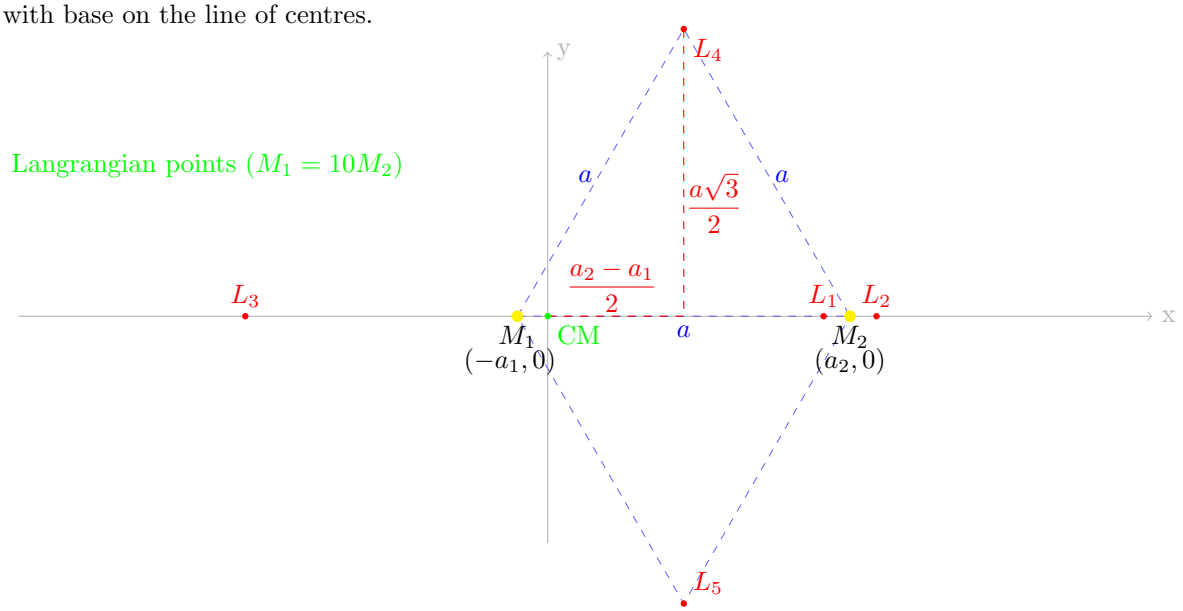
$$\frac{\frac{1}{2}a^2a_2(a_1+a_2)+\frac{1}{2}a^2a_1(-a_1-a_2)}{\left(\frac{1}{4}a^2+y^2\right)^{3/2}}-\tfrac{1}{2}(a_2-a_1)=0 \qquad (5c)$$

$$\frac{a^2a_2+a^2a_1}{\left(\frac{1}{4}a^2+y^2\right)^{3/2}}-1=0 \qquad (6c)$$

$$\frac{\frac{1}{2}a^3(a_2-a_1)}{\left(\frac{1}{4}a^2+y^2\right)^{3/2}}-\tfrac{1}{2}(a_2-a_1)=0 \qquad (5d)$$

$$\frac{a^3}{\left(\frac{1}{4}a^2+y^2\right)^{3/2}}-1=0 \qquad (6d)$$

Examining (5d) & (6d) suggests $y^2=3a^2/4$ satisfies both. Hence the other two 'equilibrium' points are at $((a_2-a_1)/2,\ a\sqrt{3}/2)$ and $((a_2-a_1)/2,\ -a\sqrt{3}/2)$. These are at the vertices of equilateral triangles with base on the line of centres.



Note: L_4 & L_5 are stable if M_1 and M_2 are sufficiently different (see Ch 12 Problem 12).