Problem

A uniform cylindrical drum of mass M and radius a is free to rotate about its axis, which is horizontal. A cable of negligible mass and length l is wound on the drum, and carries on its free end a mass m. Find the Lagrangian function if the cable is elastic, with elastic potential energy $\frac{1}{2}kx^2$, where x is the extension of the cable. Show that the motion of the mass m is a uniform acceleration of $2mg/(M+2m)$, with a superimposed oscillation of angular frequency given by $\omega^2 = k(M+2m)/Mm$. Find the amplitude of this oscillation if the system is released from rest with cable unextended.

Solution

$$\begin{aligned} L &= T - V \\ &= \left(\tfrac{1}{2}I_3\dot{\theta}^2 + \tfrac{1}{2}m(a\dot{\theta}+\dot{x})^2\right) - \left(-mg(a\theta+x) + \tfrac{1}{2}kx^2\right) \\ &= \tfrac{1}{4}Ma^2\dot{\theta}^2 + \tfrac{1}{2}m(a\dot{\theta}+\dot{x})^2 + mga\theta + mgx - \tfrac{1}{2}kx^2 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \\ \tfrac{1}{2}Ma^2\ddot{\theta} + ma(a\ddot{\theta}+\ddot{x}) &= mga \\ Ma\ddot{\theta} + 2m(a\ddot{\theta}+\ddot{x}) &= 2mg \qquad (1) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ m(a\ddot{\theta}+\ddot{x}) &= mg - kx \qquad (2) \end{aligned}$$

Solving (1) and (2) for $\ddot{x}$ and $a\ddot{\theta}$ gives

$$\ddot{x} = g - \frac{k(M+2m)}{Mm}x \qquad (3)$$

$$a\ddot{\theta} = \frac{2k}{M}x \qquad (4)$$

From (3)

$$\begin{aligned} \ddot{x} + \frac{k(M+2m)}{Mm}\left(x - \frac{gMm}{k(M+2m)}\right) &= 0 \\ \ddot{x} + \omega^2\left(x - \frac{gMm}{k(M+2m)}\right) &= 0 \qquad (\omega^2 = k(M+2m)/Mm) \\ x - \frac{gMm}{k(M+2m)} &= Acos(\omega t - \phi) \qquad (A,\ \phi \text{ constants}) \\ \dot{x} &= -A\omega\, sin(\omega t - \phi) \end{aligned}$$

Initial conditions $t = 0,\ \dot{x} = 0,\ x = 0 \quad \rightarrow \quad \phi = 0,\ A = -\dfrac{gMm}{k(M+2m)}$ hence

$$x = \frac{gMm}{k(M+2m)}(1 - cos\,\omega t)$$

Thus the total acceleration of m is given by

$$\begin{aligned} a\ddot{\theta} + \ddot{x} &= \frac{2k}{M}\frac{gMm}{k(M+2m)}\left(1 - cos\,\omega t\right) + \frac{gMm}{k(M+2m)}\omega^2 cos\,\omega t \\ &= \frac{2mg}{M+2m}\left(1 - cos\,\omega t\right) + g cos\,\omega t \\ &= \frac{2mg}{M+2m} - g cos\,\omega t\left(\frac{2m}{M+2m} - 1\right) \\ &= \frac{2mg}{M+2m} + \frac{Mg}{M+2m} cos\,\omega t \end{aligned}$$

Hence the motion has the same uniform acceleration of $2mg/(M+2m)$ as in the inelastic case, but superimposed with oscillations of angular frequency ω where $\omega^2 = k(M+2m)/Mm$

The amplitude of oscillation depends only on the second term of the acceleration. Integrating this term twice gives the oscillatory term in the displacement $(a\theta + x)$ as $-\dfrac{Mg}{(M+2m)\omega^2}\,cos\,\omega t$

Hence the amplitude of oscillation is $\dfrac{Mg}{M+2m} \times \dfrac{Mm}{k(M+2m)}$ or

$$\frac{M^2 mg}{k(M+2m)^2}$$
