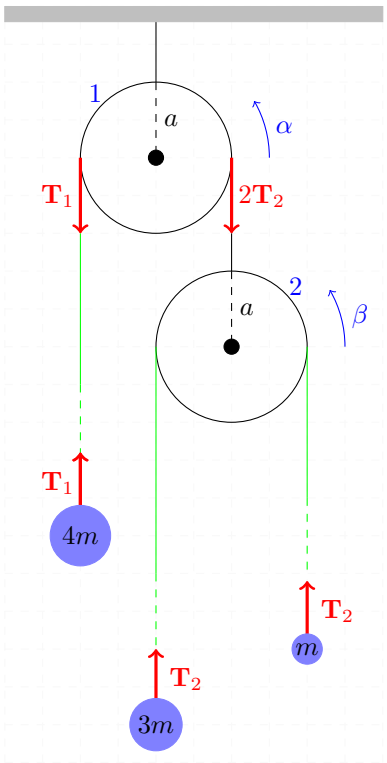
Problem

A light inextensible string passes over a light smooth pulley, and carries a mass of $4m$ on one end. The other end supports a second pulley with a string over it carrying masses $3m$ and m on the two ends. Using a suitable pair of generalized co-ordinates, write down the Lagrangian function for the system, and Lagrange's equations. Find the downward accelerations of the three masses and also the tensions in the strings. Explain why the first pulley turns, although the total mass on each side is the same.

Solution

Taking α and β as the angular displacement of pulleys 1 and 2 respectively and assuming pulleys have negligible mass



$$T = \tfrac{1}{2}(4m)(a\dot\alpha)^2 + \tfrac{1}{2}(3m)(a\dot\beta - a\dot\alpha)^2 + \tfrac{1}{2}(m)(a\dot\beta + a\dot\alpha)^2$$
$$V = -4mga\alpha - 3mga(\beta - \alpha) + mga(\beta + \alpha)$$
$$L = \frac{ma^2}{2}\left(4\dot\alpha^2 + 3(\dot\beta - \dot\alpha)^2 + (\dot\beta + \dot\alpha)^2\right) - mga(-2\beta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\alpha}\right) = \frac{\partial L}{\partial \alpha}$$
$$ma^2(8\ddot\alpha - 2\ddot\beta) = 0$$
$$4\ddot\alpha = \ddot\beta \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\beta}\right) = \frac{\partial L}{\partial \beta}$$
$$ma^2(-2\ddot\alpha + 4\ddot\beta) = 2mga$$
$$a(-\ddot\alpha + 2\ddot\beta) = g \qquad (2)$$

Subs for $\ddot\beta$ in (2) gives $\quad a\ddot\alpha = g/7 \quad \rightarrow \quad a\ddot\beta = 4g/7$

Hence the *downward* accelerations

$a\ddot\alpha$	$g/7$	for the mass $4m$
$a(\ddot\beta - \ddot\alpha)$	$3g/7$	for the mass $3m$
$a(-\ddot\beta - \ddot\alpha)$	$-5g/7$	for the mass m i.e. upwards

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The tension in the strings is given by

$$4mg - T_1 = (4m)g/7 \qquad \rightarrow \quad T_1 = 24mg/7$$
$$3mg - T_2 = (3m)3g/7 \qquad \rightarrow \quad T_2 = 12mg/7$$

If both pulleys are released simultaneously from rest, the second one will begin to turn reducing the string tension on the right hand side of the first pulley (from $4mg$ towards $24mg/7$) and hence the first pulley will begin to turn.