Problem

A top consists of a uniform, solid cone of height $50\,mm$ and base radius $20\ mm$. It is spinning with its vertex fixed at $7200\,r.p.m.$ Evaluate accurately the two possible precessional angular velocities of the top, if the axis makes an angle of 30° with the vertical. Compare the slower value with the result found earlier (Chapter 9, Problem 13 the approximate value was calculated as $4.065\,s^{-1}$). Find also the minimum angular velocity ω_3 for which steady precession at this angle is possible.

Solution

From §10.3 in the text, the precessional angular velocities Ω about the vertical satisfy the equation:

$$I_1 cos\theta\,\Omega^2 - I_3\omega_3\,\Omega + MgR = 0$$

The principal moments of inertia of a uniform solid cone about its vertex were derived in Chapter 9, Problem 12, $I_1 = 3M(a^2+4h^2)/20$ and $I_3 = 3Ma^2/10$. R is the distance from the vertex to the center of mass. Hence

$$I_1 = 3M(0.02^2 + 4\times 0.05^2)/20 = 1.56\times 10^{-3}\,M \qquad I_3 = 3M(0.02^2)/10 = 1.2\times 10^{-4}\,M$$

$$\omega_3 = 7200/60\times 2\pi = 240\pi \qquad R = 3h/4 = 3.75\times 10^{-2}$$

Solving the above equation for Ω

$$\begin{aligned}
\Omega &= \frac{I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4I_1MgR\,cos\theta}}{2I_1 cos\theta} \\
&= \frac{(1.2\times 10^{-4})(240\pi) \pm \sqrt{(1.2\times 10^{-4})^2(240\pi)^2 - 4(1.56\times 10^{-3})(9.807)(3.75\times 10^{-2})cos30^\circ}}{2(1.56\times 10^{-3})cos30^\circ} \\
&= \frac{9.048\times 10^{-2} \pm \sqrt{(8.186\times 10^{-3}) - (1.987\times 10^{-3})}}{2.702\times 10^{-3}} \\
&= \frac{(9.048\times 10^{-2}) \pm (7.873\times 10^{-2})}{2.702\times 10^{-3}} \\
\Omega_+ &= \frac{16.921\times 10^{-2}}{2.702\times 10^{-3}} = 62.62\,s^{-1} \qquad \& \qquad \Omega_- = \frac{1.175\times 10^{-2}}{2.702\times 10^{-3}} = 4.349\,s^{-1}
\end{aligned}$$

Thus the approximate value of $4.065\,s^{-1}$ derived in Chapter 9 by assuming that $\Omega << \omega$ compares reasonably well with the more accurate value Ω_-.

For steady precession Ω must be real, hence the minimum angular velocity possible is given by

$$\begin{aligned}
I_3^2\omega_{min}^2 &= 4I_1MgR\,cos\theta \\
\omega_{min}^2 &= \frac{4I_1MgR\,cos\theta}{I_3^2} \\
&= \frac{4(1.56\times 10^{-3})(9.807)(3.75\times 10^{-2})\,cos30^\circ}{(1.2\times 10^{-4})^2} = \frac{1.987\times 10^{-3}}{1.44\times 10^{-8}} \\
&= 1.380\times 10^5 \\
\omega_{min} &= 371.5\,s^{-1}
\end{aligned}$$

The minimum required angular velocity is $60\times 371.5/2\pi$ or $3548\,r.p.m.$