Problem

A simple pendulum of mass m and length l hangs from a trolley of mass M running on smooth horizontal rails. Using the position x of the trolley and the angle of inclination θ of the pendulum as generalized co-ordinates, write down the Lagrangian function, and Lagrange's equations. Obtain an equation of motion for θ alone. If the system is released from rest with the pendulum inclined at 30° to the vertical, use energy conservation to find its angular velocity when it reaches the vertical, given that $M = 2\,kg$, $m = 1\,kg$, and $l = 2\,m$.

Solution

$$\begin{aligned} L &= \tfrac{1}{2}M\dot{x}^2 + \left(\tfrac{1}{2}m(\dot{x} + l\dot{\theta}cos\theta)^2 + \tfrac{1}{2}ml^2\dot{\theta}^2sin^2\theta\right) - (mgl(1 - cos\theta)) \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m(\dot{x}^2 + 2\dot{x}l\dot{\theta}cos\theta + l^2\dot{\theta}^2) - mgl(1 - cos\theta) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \frac{d}{dt}(M\dot{x} + m\dot{x} + ml\dot{\theta}cos\theta) &= 0 \\ (M + m)\ddot{x} + ml(\ddot{\theta}cos\theta - \dot{\theta}^2sin\theta) &= 0 \qquad (1) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \\ \frac{d}{dt}(mlcos\theta\dot{x} + ml^2\dot{\theta}) &= -ml\dot{x}\dot{\theta}sin\theta - mglsin\theta \\ \ddot{x}cos\theta - \dot{x}\dot{\theta}sin\theta + l\ddot{\theta} &= -\dot{x}\dot{\theta}sin\theta - gsin\theta \\ \ddot{x}cos\theta &= -l\ddot{\theta} - gsin\theta \qquad (2) \end{aligned}$$

Then subs for $\ddot{x}$ in (1)

$$\begin{aligned} -(M + m)(l\ddot{\theta} + gsin\theta) + mlcos\theta(\ddot{\theta}cos\theta - \dot{\theta}^2sin\theta) &= 0 \\ -(M + m)(l\ddot{\theta} + gsin\theta) + mlcos^2\theta\,\ddot{\theta} - mlsin\theta\cos\theta\,\dot{\theta}^2 &= 0 \\ -l\ddot{\theta}(M + m - m(1 - sin^2\theta)) - mlsin\theta\cos\theta\,\dot{\theta}^2 - (M + m)gsin\theta &= 0 \\ l\ddot{\theta}(M + msin^2\theta)) + mlsin\theta\cos\theta\,\dot{\theta}^2 + (M + m)gsin\theta &= 0 \end{aligned}$$

Equating the kinetic energy at the vertical with the initial potential energy

$$\tfrac{1}{2}(M + m)\dot{x}_1^2 + ml\dot{x}_1\dot{\theta}_1 + \tfrac{1}{2}ml^2\dot{\theta}_1^2 = mgl(1 - cos30^\circ)$$

Subs $M = 2$, $m = 1$, $l = 2$

$$\tfrac{3}{2}\dot{x}_1^2 + 2\dot{x}_1\dot{\theta}_1 + 2\dot{\theta}_1^2 = 2g(1 - cos30^\circ)$$

Taking the momenta of the two masses as equal and opposite at the vertical i.e. $m(\dot{x}_1 + l\dot{\theta}_1) = -M\dot{x}_1$ or $\dot{x}_1 = -2\dot{\theta}_1/3$ and substituting

$$\begin{aligned} \dot{\theta}_1^2(\frac{2}{3} - \frac{4}{3} + 2) &= 2g(1 - cos30^\circ) \\ \dot{\theta}_1^2 &= 3g(1 - cos30^\circ)/2 = 1.971 \\ \dot{\theta}_1 &= 1.403 \end{aligned}$$

The angular velocity at the vertical is $1.40\,s^{-1}$