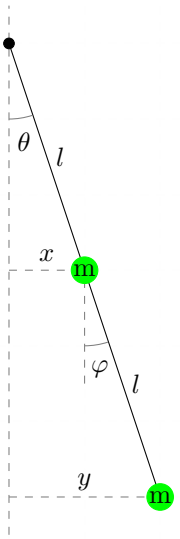
Problem

A double pendulum, consisting of a pair, each of mass m and length l, is released from rest with the pendulums displaced but in a straight line. Find the displacements of the pendulums as functions of time.



Note: Diagram shows conditions at $t = 0$ i.e $\theta = \varphi$. For $t > 0$ this is not the case.

Solution

For small oscillations, using generalized coords x and y

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 \\ V &= 2mgl(1 - cos\theta) + mgl(1 - cos\varphi) \\ &= mgl\theta^2 + \tfrac{1}{2}mgl\varphi^2 \\ &= \frac{mg}{l}x^2 + \frac{mg}{2l}(y - x)^2 \\ L &= T - V \\ &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 - \frac{mg}{l}x^2 - \frac{mg}{2l}(y - x)^2 \end{aligned}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -3g/l & g/l \\ g/l & -g/l \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The solutions to these equations are of the form $q_\alpha = A_\alpha e^{i\omega t}$, hence subs for x and y yields the *normal mode* equations:

$$\begin{pmatrix} 3g/l & -g/l \\ -g/l & g/l \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \omega^2 \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$\begin{pmatrix} 3g/l - \omega^2 & -g/l \\ -g/l & g/l - \omega^2 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These equations have a non-zero solution if and only if the determinant of the coefficients is zero. Hence the *characteristic equation* for the system and normal mode frequencies ($\omega_\pm$) are given by

$$\begin{aligned} (3g/l - \omega^2)(g/l - \omega^2) - g^2/l^2 &= 0 \\ \omega^4 - \frac{4g}{l}\omega^2 + \frac{2g^2}{l^2} &= 0 \\ \omega^2 &= \frac{4g/l \pm \sqrt{16g^2/l^2 - 8g^2/l^2}}{2} \\ \omega_\pm^2 &= (2 \pm \sqrt{2})g/l \end{aligned}$$

Subs these *eigenvalues* in the normal mode equations gives the ratio of amplitudes for each mode

$$\begin{aligned} \omega_+^2 = (2+\sqrt{2})g/l \quad &\rightarrow \quad 3A_x^+ - A_y^+ = (2+\sqrt{2})A_x^+ \quad \rightarrow \quad A_y^+ = (1-\sqrt{2})A_x^+ \\ \omega_-^2 = (2-\sqrt{2})g/l \quad &\rightarrow \quad 3A_x^+ - A_y^- = (2-\sqrt{2})A_x^- \quad \rightarrow \quad A_y^- = (1+\sqrt{2})A_x^- \end{aligned}$$

The general solutions are then the real part of the superposition of the normal modes

$$\begin{aligned} x &= A_x^+ e^{i\omega_+ t} + A_x^- e^{i\omega_- t} \\ y &= A_y^+ e^{i\omega_+ t} + A_y^- e^{i\omega_- t} \end{aligned}$$

Using initial conditions $t = 0 \quad x = a \quad y = 2a$ and subs for A_y

$$\begin{aligned} a &= A_x^+ + A_x^- \\ 2a &= A_x^+(1-\sqrt{2}) + A_x^-(1+\sqrt{2}) \end{aligned}$$

Solving these equations yields

$$A_x^+ = \frac{a}{2}(1-\sqrt{2}/2) \qquad A_x^- = \frac{a}{2}(1+\sqrt{2}/2)$$

And hence

$$A_y^+ = \frac{a}{2}(2-3\sqrt{2}/2) \qquad A_y^- = \frac{a}{2}(2+3\sqrt{2}/2)$$

Thus the general solutions are

$$\begin{aligned} x &= Re\left[\frac{a}{2}(1-\sqrt{2}/2)e^{i\omega_+ t} + \frac{a}{2}(1+\sqrt{2}/2)e^{i\omega_- t}\right] \\ &= \frac{a}{2}\left((cos\omega_+ t + cos\omega_- t) - \frac{\sqrt{2}}{2}\,(cos\omega_+ t - cos\omega_- t)\right) \\ &= a(cos\omega_1 t\, cos\omega_2 t + \frac{\sqrt{2}}{2}\, sin\omega_1\, t sin\omega_2 t) \end{aligned}$$

$$\begin{aligned} y &= Re\left[\frac{a}{2}(2-3\sqrt{2}/2)e^{i\omega_+ t} + \frac{a}{2}(2+3\sqrt{2}/2)e^{i\omega_- t}\right] \\ &= \frac{a}{2}\left(2(cos\omega_+ t + cos\omega_- t) - \frac{3\sqrt{2}}{2}\,(cos\omega_+ t - cos\omega_- t)\right) \\ &= a(2cos\omega_1 t\, cos\omega_2 t + \frac{3\sqrt{2}}{2}\, sin\omega_1\, t sin\omega_2 t) \end{aligned}$$

Where $\omega_{1,2} = \frac{1}{2}(\omega_+ \pm \omega_-)$
