Problem

A rigid rod of length $2a$ is suspended by two light, inextensible strings of length l joining its ends to supports also a distance $2a$ apart and level with each other. Using the longitudinal displacement x, and the tranverse displacements y_1, y_2 of its ends, as generalized co-ordinates, find the Lagrangian function (for small x, y_1, y_2). Determine the normal modes and frequencies.

Solution

Taking the co-ordinates of the CM of the rod as (x,y) and the angle through which it rotates about the CM as θ

$$y = \frac{y_1 + y_2}{2} \qquad\qquad \theta = \frac{y_1 - y_2}{2a}$$

The longitudinal, transverse and rotational modes all contribute to the potential energy, hence for small oscillations (using $\omega_0^2 = g/l$)

$$\begin{aligned} V &= \tfrac{1}{2}m\omega_0^2 x^2 + \tfrac{1}{2}m\omega_0^2 y^2 + mg\left(l - \sqrt{l^2 - \left(\frac{y_1 - y_2}{2}\right)^2}\right) \\ &= \tfrac{1}{2}m\omega_0^2 x^2 + \tfrac{1}{8}m\omega_0^2 (y_1 + y_2)^2 + mgl\left(1 - \left(1 - (y_1 - y_2)/2l\right)^2\right)^{1/2}\Big) \\ &\approx \tfrac{1}{2}m\omega_0^2 x^2 + \tfrac{1}{8}m\omega_0^2 (y_1 + y_2)^2 + mgl(y_1 - y_2)^2/8l^2 \\ &= \tfrac{1}{2}m\omega_0^2 x^2 + \tfrac{1}{8}m\omega_0^2 (y_1 + y_2)^2 + \tfrac{1}{8}m\omega_0^2 (y_1 - y_2)^2 \end{aligned}$$

Similarly for the kinetic energy

$$\begin{aligned} T &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}I\dot{\theta}^2 \\ &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{8}m(\dot{y}_1 + \dot{y}_2)^2 + \tfrac{1}{2}\left(\tfrac{1}{3}ma^2\right)(\dot{y}_1 - \dot{y}_2)^2/4a^2 \\ &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{8}m(\dot{y}_1 + \dot{y}_2)^2 + \tfrac{1}{24}m(\dot{y}_1 - \dot{y}_2)^2 \end{aligned}$$

Hence the Lagrangian

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{8}m(\dot{y}_1 + \dot{y}_2)^2 + \tfrac{1}{24}m(\dot{y}_1 - \dot{y}_2)^2 - \tfrac{1}{2}m\omega_0^2 x^2 - \tfrac{1}{8}m\omega_0^2 (y_1 + y_2)^2 - \tfrac{1}{8}m\omega_0^2 (y_1 - y_2)^2$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \ddot{x} &= -\omega_0^2 x \qquad\qquad \rightarrow \qquad\qquad \omega_x^2 = \omega_0^2 = g/l \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) &= \frac{\partial L}{\partial y_1} \\ \tfrac{1}{4}m(\ddot{y}_1 + \ddot{y}_2) + \tfrac{1}{12}m(\ddot{y}_1 - \ddot{y}_2) &= -\tfrac{1}{4}m\omega_0^2(y_1 + y_2) - \tfrac{1}{4}m\omega_0^2(y_1 - y_2) \\ 2\ddot{y}_1 + \ddot{y}_2 &= -3\omega_0^2 y_1 \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_2}\right) &= \frac{\partial L}{\partial y_2} \\ \tfrac{1}{4}m(\ddot{y}_1 + \ddot{y}_2) - \tfrac{1}{12}m(\ddot{y}_1 - \ddot{y}_2) &= -\tfrac{1}{4}m\omega_0^2(y_1 + y_2) + \tfrac{1}{4}m\omega_0^2(y_1 - y_2) \\ \ddot{y}_1 + 2\ddot{y}_2 &= -3\omega_0^2 y_2 \end{aligned} \tag{2}$$

Adding or subtracting (1) and (2) gives the required equations for y and θ

$$\begin{aligned} \ddot{y}_1 + \ddot{y}_2 &= -\omega_0^2(y_1 + y_2) \\ \ddot{y} &= -\omega_0^2 y \qquad \rightarrow \qquad \omega_y^2 = \omega_0^2 = g/l \end{aligned}$$

$$\begin{aligned} \ddot{y}_1 - \ddot{y}_2 &= -3\omega_0^2(y_1 - y_2) \\ \ddot{\theta} &= -3\omega_0^2\theta \qquad \rightarrow \qquad \omega_\theta^2 = 3\omega_0^2 = 3g/l \end{aligned}$$

Hence the normal mode frequencies are $\sqrt{\frac{g}{l}}$, $\sqrt{\frac{g}{l}}$ and $\sqrt{\frac{3g}{l}}$.
