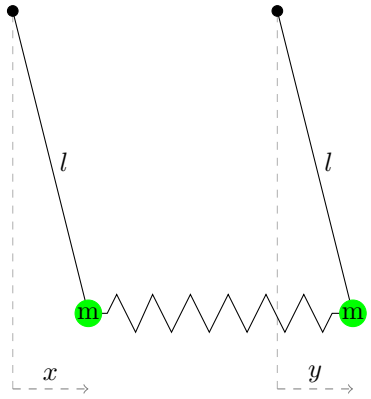
Problem

In a pair of coupled pendulums, each pendulum is subjected to a damping force, of magnitude $\alpha\dot{x}$ and $\alpha\dot{y}$ respectively, while there is a damping force $\beta(\dot{x}-\dot{y})$ in the spring. Show that he equations of motion for the normal co-ordinates q_1 and q_2 are still uncoupled. Find the amplitudes of the forced oscillations obtained by applying a periodic force to one pendulum. Given that the forcing frequency is that of the uncoupled pendulums, and that β is negligible, find the range of values of α for which the amplitude of the second pendulum is less than half that of the first.

Solution



With *no damping* the general solution for a pair of coupled pendulums is the real part of a superposition of the two normal modes (see §11.4 in the text)

$$x = Ae^{i\omega_0 t} + A'e^{i\omega' t}$$
$$y = Ae^{i\omega_0 t} - A'e^{i\omega' t}$$

$\omega_0^2 = g/l \quad$ and $\quad \omega'^2 = \omega_0^2 + 2\omega_s^2 \quad (\omega_s^2 = k/m)$

A and A' are determined by the initial conditions.

The amplitude A is subject to the damping of the pendulum and the amplitude A' to both the damping of pendulum and spring, hence

$$x = Ae^{i\omega_0 t-\gamma t} + A'e^{i\omega' t-\gamma' t}$$
$$y = Ae^{i\omega_0 t-\gamma t} - A'e^{i\omega' t-\gamma' t}$$

Where $\gamma = \alpha/2m \qquad$ and $\quad \gamma' = (\alpha+2\beta)/2m \quad$ (see §2.5 in the text)

Introducing the normal co-ordinates

$$q_1 = \sqrt{\frac{m}{2}}(x+y) \qquad\qquad q_2 = \sqrt{\frac{m}{2}}(x-y)$$

the general solutions become

$$q_1 = A_1 e^{i\omega_0 t-\gamma t} \qquad A_1 = \sqrt{2m}\,A$$
$$q_2 = A_2 e^{i\omega' t-\gamma' t} \qquad A_2 = \sqrt{2m}\,A'$$

Hence in these co-ordinates q_1 and q_2 are still uncoupled.

Taking the forcing frequency as the real part of $Fe^{i\omega_1 t}$, the equations of motion are

$$\ddot{q}_1 + 2\gamma\dot{q}_1 + \omega_0^2 q_1 = Fe^{i\omega_1 t}/\sqrt{2m} \qquad\qquad \ddot{q}_2 + 2\gamma'\dot{q}_2 + \omega'^2 q_2 = Fe^{i\omega_1 t}/\sqrt{2m}$$

These independent oscillator equations are solved by looking for solutions periodic in time with the same frequency as the applied force (see §2.6 in the text). This yields

$$(\omega_0^2 + 2i\gamma\omega_1 - \omega_1^2)A_1 = F/\sqrt{2m} \qquad\qquad (\omega'^2 + 2i\gamma'\omega_1 - \omega_1^2)A_2 = F/\sqrt{2m}$$

Note: $A = ae^{-i\theta}$ is complex.

Hence the complex amplitudes of the forced oscillations are

$$A_1 = F/\left(\sqrt{2m}(\omega_0^2 - \omega_1^2 + 2i\gamma\omega_1)\right) \qquad A_2 = F/\left(\sqrt{2m}(\omega'^2 - \omega_1^2 + 2i\gamma'\omega_1)\right)$$

$\omega_0^2 = g/l \qquad \omega'^2 = \omega_0^2 + 2\omega_s^2 \quad (\omega_s^2 = k/m) \qquad \omega_1 =$ forcing frequency

$\gamma = \alpha/2m \qquad \gamma' = (\alpha + 2\beta)/2m$

The actual amplitudes of q_1 and q_2 are

$$a_1 = \frac{F/\sqrt{2m}}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} \qquad a_2 = \frac{F/\sqrt{2m}}{\sqrt{(\omega'^2 - \omega_1^2)^2 + 4\gamma'^2\omega_1^2}}$$

As shown in §2.6 (or taking the root of AA^*)

Given $\omega_1 = \omega_0$ and $\gamma' = \gamma$

$$A_1 = F/\left(\sqrt{2m}(2i\gamma\omega_0)\right) \qquad A_2 = F/\left(\sqrt{2m}(2\omega_s^2 + 2i\gamma\omega_0)\right)$$

The amplitudes of x and y are given in the text as

$$A_x = \frac{A_1 + A_2}{\sqrt{2m}} \qquad A_y = \frac{A_1 - A_2}{\sqrt{2m}}$$

Hence

$$\begin{aligned}
\frac{A_x}{A_y} &= \frac{A_1 + A_2}{A_1 - A_2} \\
&= \frac{1/2i\gamma\omega_0 + 1/(2\omega_s^2 + 2i\gamma\omega_0)}{1/2i\gamma\omega_0 - 1/(2\omega_s^2 + 2i\gamma\omega_0)} \\
&= \frac{2\omega_s^2 + 4i\gamma\omega_0}{2\omega_s^2} \\
&= 1 + \frac{i\alpha\omega_0}{k} \\
\frac{a_x}{a_y} &= \sqrt{1 + \frac{\alpha^2\omega_0^2}{k^2}}
\end{aligned}$$

For $a_x : a_y > 2 : 1$

$$\begin{aligned}
\frac{\alpha^2\omega_0^2}{k^2} &> 3 \\
\alpha &> \sqrt{3}k/\omega_0
\end{aligned}$$
