Problem

Show that a stretched string is equivalent mathematically to an infinite number of uncoupled oscillators, described by the co-ordinates

$$q_n(t) = \sqrt{\frac{2}{l}} \int_0^l y(x,t) sin\frac{n\pi x}{l} dx\,.$$

A string of length l, fixed at both ends, is released from rest with its mid-point displaced a distance a, and each half of the string straight, determine the amplitudes of the various normal modes. Why, physically, are the modes for even values of n not excited?

Solution

The general solution for a stretched string is a superposition of normal modes which may be expanded as a Fourier series *

$$y(x,t) = \sum_{n=1}^{\infty} A_n(t) sin\frac{n\pi x}{l}$$

Using this expression to derive the kinetic energy

$$\begin{aligned} T &= \int_0^l \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 \quad \text{(where } \mu \text{ is mass/unit length)} \\ &= \tfrac{1}{2}\mu \int_0^l \left(\sum_{m=1}^{\infty} \dot{A}_m(t) sin\frac{m\pi x}{l}\right)\left(\sum_{n=1}^{\infty} \dot{A}_n(t) sin\frac{n\pi x}{l}\right) \\ &= \tfrac{1}{2}\mu \sum_{n=1}^{\infty} \left(\dot{A}_n(t)\right)^2 \frac{l}{2} \quad ** \\ &= \tfrac{1}{2}\mu \sum_{n=1}^{\infty} \left(\sqrt{\frac{l}{2}}\,\dot{A}_n(t)\right)^2 \end{aligned}$$

In Lagrangian mechanics, using $q_n(t)$ as generalised vertical coordinate $T = \frac{1}{2}\mu \sum_{n=1}^{\infty} \dot{q_n}^2$. Comparing these two expression for T gives $A_n(t) = \sqrt{\frac{2}{l}}\; q_n(t)$

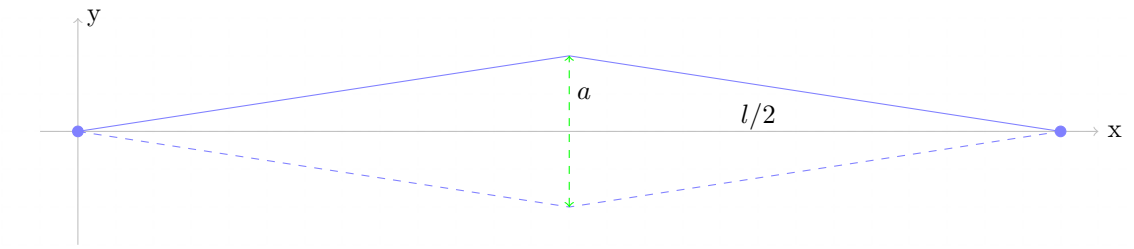
Substituting for A_n in the original equation for $y(x,t)$ gives

$$y(x,t) = \sum_{m=1}^{\infty} \sqrt{\frac{2}{l}}\; q_m(t) sin\frac{m\pi x}{l}$$

Multiplying both sides by $sin\frac{n\pi x}{l}$ and integrating w.r.t. x

$$\begin{aligned} \int_0^l y(x,t) sin\frac{n\pi x}{l}\, dx &= \sum_{m=1}^{\infty} \sqrt{\frac{2}{l}}\; q_m(t) \int_0^l sin\frac{m\pi x}{l} sin\frac{n\pi x}{l}\, dx \\ &= \sqrt{\frac{2}{l}}\; q_n(t)\, \frac{l}{2} \quad ** \\ q_n(t) &= \sqrt{\frac{2}{l}} \int_0^l y(x,t) sin\frac{n\pi x}{l}\, dx \end{aligned}$$

For a string length l with mid-point displaced by a



$$y(x) = \begin{cases} 2ax/l & 0 \le x \le l/2 \\ 2a - 2ax/l & l/2 \le x \le l \end{cases}$$

Coordinates of normal modes

$$q_n = \sqrt{\frac{2}{l}} \int_0^l y(x) sin \frac{n\pi x}{l} dx$$

In this case only the odd modes are required, hence

$$\begin{aligned} q_{2r+1} &= \sqrt{\frac{2}{l}} \int_0^l y(x) sin \frac{(2r+1)\pi x}{l} dx \\ &= 2\sqrt{\frac{2}{l}} \int_0^{l/2} \frac{2ax}{l} sin \frac{(2r+1)\pi x}{l} dx \\ &= \frac{4a}{l}\sqrt{\frac{2}{l}} \int_0^{l/2} x sin \frac{(2r+1)\pi x}{l} dx \\ &= \frac{4a}{l}\sqrt{\frac{2}{l}} \left[\frac{sin\big((2r+1)\pi x/l\big)}{(2r+1)^2\pi^2/l^2} - \frac{x cos\big((2r+1)\pi x/l\big)}{(2r+1)\pi/l} \right]_0^{l/2} \\ &= 4a\sqrt{2l}\ \frac{sin\big((2r+1)\pi/2\big)}{(2r+1)^2\pi^2} \\ &= (-1)^r \frac{4a\sqrt{2l}}{(2r+1)^2\pi^2} \end{aligned}$$

For even modes $y = 0$ at the mid-point and they do not contribute to the standing wave.

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* Kibble expresses this in more detail at the end of the chapter

$$y(x,t) = 2Re \sum_{n=1}^{\infty} A_n e^{in\pi ct/l} sin \frac{n\pi x}{l} dx$$

but for the purpose of this derivation it's unecessary.

** This result relies on the orthogonality of sines, the integral vanishes except for $m = n$ which returns a value of $l/2$.