Problem

For a light string length $(n+1)l$, tension F, with n equal masses m placed at regular intervals l, the normal mode equations are (see eq 11.33 in the text)

$$\begin{pmatrix} 2\omega_0^2-\omega^2 & -\omega_0^2 & 0 & ... & 0 \\ -\omega_0^2 & 2\omega_0^2-\omega^2 & -\omega_0^2 & ... & 0 \\ 0 & -\omega_0 & 2\omega_0^2-\omega^2 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & 2\omega_0^2-\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ A_n \end{pmatrix} = 0$$

Show that a typical equation in the set may be satisfied by setting $A_\alpha = sin\alpha k \, (\alpha = 1, 2, ..., n)$ provided that $\omega = 2\omega_0 sin\frac{1}{2}k$. Hence show by considering the required condition when $\alpha = n+1$ that the frequencies of the normal modes are $\omega_r = 2\omega_0 sin\big(r\pi/2(n+1)\big)$ with $r = 1, 2, ..., n$. Why may we ignore values of r greater than $n+1$? Show that, in the limit of large n, the frequency of the r^{th} normal mode tends to the corresponding frequency of the continuous string with the same total length and mass.

Solution

For a typical particle α (using $\omega_0^2 = F/ml$)

$$\begin{aligned} -\omega_0^2 A_{\alpha-1} + (2\omega_0^2 - \omega^2)A_\alpha - \omega^2 A_{\alpha+1} &= 0 \\ \omega^2 A_\alpha + \omega^2(A_{\alpha-1} - 2A_\alpha + A_{\alpha+1}) &= 0 \end{aligned}$$

Subs $A_\alpha = sin\alpha k \, (\alpha = 1, 2, ..., n)$ and using trig identities gives

$$\begin{aligned} \omega^2 sin\alpha k + \omega_0^2(sin\alpha(k-1) - 2sin\alpha k + sin\alpha(k+1)) &= 0 \\ \omega^2 sin\alpha k - 2\omega_0^2(sin\alpha k - sin\alpha k cosk) &= 0 \\ \omega^2 &= 2\omega_0^2(1-cosk) \\ \omega &= 2\omega_0 sin(k/2) \end{aligned}$$

The boundary condition $A_{n+1} = sin\big((n+1)k\big) = 0 \quad \rightarrow \quad (n+1)k = n\pi$ or $k = n\pi/(n+1)$ hence the normal mode frequencies are

$$\omega_r = 2\omega_0 sin\left(\frac{r\pi}{2(n+1)}\right) \qquad \text{where} \quad r = 1, 2, ..., n$$

The number of modes r is equal to the number of particles n, hence $r > n$ is meaningless.

For large n

$$\omega_r \approx 2\omega_0 \frac{\pi}{2n} r = \pi\sqrt{\frac{F}{nmnl}}\, r$$

As $n \rightarrow \infty \quad nm \rightarrow M$ and $nl \rightarrow L \quad$ where M and L are total mass/length of the string respectively.

$$\omega_r = \pi\sqrt{\frac{F}{ML}}\, r = \pi\omega_1 \, r$$

i.e. the frequency of the r^{th} normal mode tends to the corresponding frequency of a continuous string with the same total mass and length. (Note: this only applies to small mode numbers.)