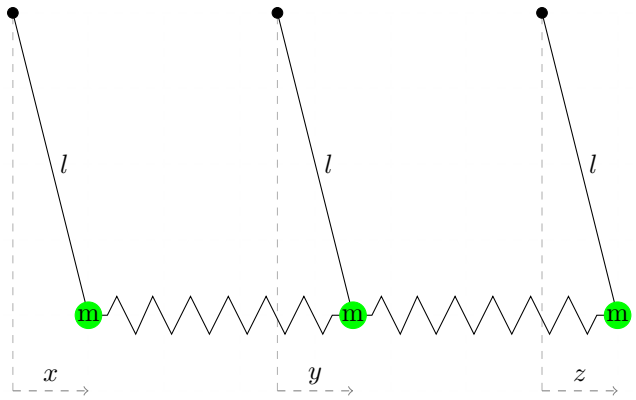
Problem

(a)Three identical pendulums are coupled with springs between the first and second and between the second and third. Find the frequencies of the normal modes, and the ratios of amplitudes. (b) The first of the three pendulums is initially displaced a distance a, while the other two are vertical. The system is released from rest. Find the maximum amplitudes of the second and third pendulums in the subsequent motion.

Solution



a) For small oscillations

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$V = \tfrac{1}{2}m\omega_0^2(x^2 + y^2 + z^2) + \tfrac{1}{2}k(x-y)^2 + \tfrac{1}{2}k(y-z)^2 \qquad (\omega_0^2 = g/l)$$

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \tfrac{1}{2}m\omega_0^2(x^2 + y^2 + z^2) - \tfrac{1}{2}k(x-y)^2 - \tfrac{1}{2}k(y-z)^2$$

For convenience, taking $\omega_s^2 = k/m$ the equations of motion are

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \begin{pmatrix}-\omega_0^2-\omega_s^2 & \omega_s^2 & 0\\ \omega_s^2 & -\omega_0^2-2\omega_s^2 & \omega_s^2\\ 0 & \omega_s^2 & -\omega_0^2-\omega_s^2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

Subs $q_\alpha = A_\alpha e^{i\omega t}$ yields the normal mode equations:

$$\begin{pmatrix}\omega_0^2+\omega_s^2 & -\omega_s^2 & 0\\ -\omega_s^2 & \omega_0^2+2\omega_s^2 & -\omega_s^2\\ 0 & -\omega_s^2 & \omega_0^2+\omega_s^2\end{pmatrix}\begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix} = \omega^2\begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix}$$

or

$$\begin{pmatrix}\omega_0^2+\omega_s^2-\omega^2 & -\omega_s^2 & 0\\ -\omega_s^2 & \omega_0^2+2\omega_s^2-\omega^2 & -\omega_s^2\\ 0 & -\omega_s^2 & \omega_0^2+\omega_s^2-\omega^2\end{pmatrix}\begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix} = 0$$

And the characteristic equation (equating the determinant of the coefficients to zero):

$$(\omega_0^2+\omega_s^2-\omega^2)\Big((\omega_0^2+2\omega_s^2-\omega^2)(\omega_0^2+\omega_s^2-\omega^2)-\omega_s^4\Big)+\omega_s^2\Big((-\omega_s^2)(\omega_0^2+\omega_s^2-\omega^2)\Big)=0$$

By inspection the three roots of this equation are:

$$\omega^2 = \omega_0^2 \qquad \omega'^2 = \omega_0^2+\omega_s^2 \qquad \omega''^2 = \omega_0^2+3\omega_s^2$$

These are easily verified by substitution.

$\omega^2 = \omega_0^2 \quad \rightarrow \quad A_x - A_y = 0 \quad \& \quad -A_x + 2A_y - A_z = 0 \qquad \rightarrow \quad A_x : A_y : A_z = 1 : 1 : 1$

$\omega'^2 = \omega_0^2 + \omega_s^2 \quad \rightarrow \quad A'_y = 0 \quad \& \quad -A'_x + A'_y - A'_z = 0 \qquad \rightarrow \quad A'_x : A'_y : A'_z = 1 : 0 : -1$

$\omega''^2 = \omega_0^2 + 3\,\omega_s^2 \quad \rightarrow \quad -2A''_x - A''_y = 0 \quad \& \quad -A''_x - A''_y - A''_z = 0 \qquad \rightarrow \quad A''_x : A''_y : A''_z = 1 : -2 : 1$

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b) The general solutions are the real part of

$$x = A_x e^{i\omega t} + A'_x e^{i\omega' t} + A''_x e^{i\omega'' t}$$
$$y = A_y e^{i\omega t} + A'_y e^{i\omega' t} + A''_y e^{i\omega'' t}$$
$$z = A_z e^{i\omega t} + A'_z e^{i\omega' t} + A''_z e^{i\omega'' t}$$

Subs $t = 0 \quad x = a \quad y = 0 \quad z = 0$

$$a = A_x + A'_x + A''_x$$
$$0 = A_y + A'_y + A''_y$$
$$0 = A_z + A'_z + A''_z$$

These equations together with the above ratios of amplitudes give:

$A_x = A_y = A_z = a/3$

$A'_x = a/2 \quad A'_y = 0 \quad A'_z = -a/2$

$A''_x = a/6 \quad A''_y = -a/3 \quad A''_z = a/6$

Hence the maximum amplitude for the second pendulum is $|A_y| + |A''_y| = 2a/3$

And for the third pendulum $|A_z| + |A'_z| + A''_z| = a$
