Problem

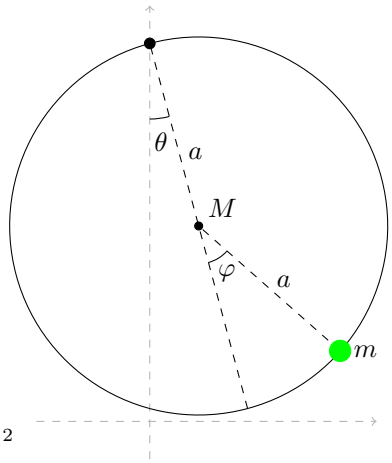
A bead of mass m slides on a smooth circular hoop of mass M and radius a, which is pivoted at a point on the rim so that it can swing freely in its plane. Write down the Lagrangian in terms of the angle θ of the diameter through the pivot and the angular position φ of the bead relative to a fixed point on the hoop. Find the frequencies of the normal modes, and sketch the configuration of the hoop and bead at the extreme point of each.

Solution

For small oscillations

$$T = \tfrac{1}{2} I\dot{\theta}^2 + \tfrac{1}{2} m\dot{x}^2$$

Where I is the moment of inertia of the hoop about the fixed point on the rim i.e. $2Ma^2$ and $x = a\big(sin\theta + sin(\theta+\varphi)\big)$



$$V = (M+m)ga(1-cos\theta) + mga\big(1-cos(\theta+\varphi)\big)$$

Hence using the small angle approximations

$$T = Ma^2\dot{\theta}^2 + \tfrac{1}{2}ma^2(2\dot{\theta}+\dot{\varphi})^2$$
$$V = \tfrac{1}{2}(M+m)ga\theta^2 + \tfrac{1}{2}mga(\theta+\varphi)^2$$

$$L = Ma^2\dot{\theta}^2 + \tfrac{1}{2}ma^2(2\dot{\theta}+\dot{\varphi})^2 - \tfrac{1}{2}(M+m)ga\theta^2 - \tfrac{1}{2}mga(\theta+\varphi)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
$$2Ma^2\ddot{\theta} + 2ma^2(2\ddot{\theta}+\ddot{\varphi}) = -(M+m)ga\theta - mga(\theta+\varphi)$$
$$2(M+2m)a\ddot{\theta} + 2ma\ddot{\varphi} = -(M+2m)g\theta - mg\varphi \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}$$
$$ma^2(2\ddot{\theta}+\ddot{\varphi}) = -mga(\theta+\varphi)$$
$$2a\ddot{\theta} + a\ddot{\varphi} = -g\theta - g\varphi \qquad (2)$$

$(1) - 2m \times (2) \quad \rightarrow \quad 2Ma\ddot{\theta} = -Mg\theta + mg\varphi \quad \text{or} \quad \ddot{\theta} = -\dfrac{g}{2a}\,\theta + \dfrac{m}{2M}\dfrac{g}{a}\,\varphi$

Subs for $\ddot{\theta}$ in (2) $\quad \rightarrow \quad a\ddot{\varphi} = -g\theta - g\varphi - 2a\left(-\dfrac{g}{2a}\,\theta + \dfrac{m}{2M}\dfrac{g}{a}\,\varphi\right) \quad \text{or} \quad \ddot{\varphi} = -\dfrac{M+m}{M}\dfrac{g}{a}\,\varphi$

The equations of motion are

$$\begin{pmatrix}\ddot{\theta}\\ \ddot{\varphi}\end{pmatrix} = \begin{pmatrix}-g/2a & mg/2Ma\\ 0 & -(M+m)g/Ma\end{pmatrix}\begin{pmatrix}\theta\\ \varphi\end{pmatrix}$$

Hence the normal mode equations (by subs $q_\alpha = A_\alpha e^{i\omega t}$)

$$\begin{pmatrix}g/2a-\omega^2 & -mg/2Ma\\ 0 & (M+m)g/Ma-\omega^2\end{pmatrix}\begin{pmatrix}A_\theta\\ A_\varphi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

The frequencies of the normal modes are given by equating the determinant of the coefficients to zero:

$$(g/2a - \omega^2)\left((M+m)g/Ma - \omega^2\right) = 0$$

Hence

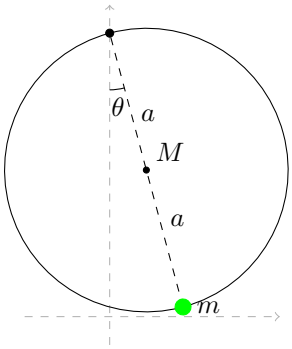
$$\omega^2 = \frac{g}{2a} \qquad\qquad \omega'^{\,2} = \frac{M+m}{M}\frac{g}{a} \quad *$$

$$\omega^2 = \frac{g}{2a} \quad \rightarrow \quad A_\varphi = 0$$

$$\omega'^{\,2} = \frac{M+m}{M}\frac{g}{a} \quad \rightarrow \quad \left(\frac{g}{2a} - \frac{M+m}{M}\frac{g}{a}\right)A'_\theta - \frac{m}{2M}\frac{g}{a}A'_\varphi = 0 \quad \rightarrow \quad A'_\theta/A'_\varphi = -1$$

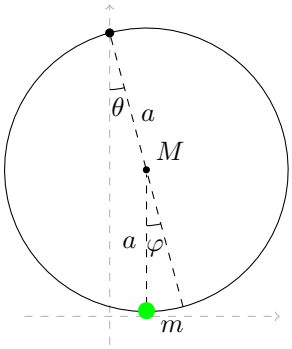
The general solutions are the real part of a superposition of the normal modes

$$\theta = A_\theta e^{i\omega t} - A' e^{i\omega' t}$$
$$\varphi = A' e^{i\omega' t}$$



If $\omega' = 0$ the bead remains opposite the pivot and the system behaves like a simple pendulum of length $2a$ with $\omega = \sqrt{g/2a}$.

If $\omega = 0$ then the angles θ and φ are equal in magnitude, but the rotations are in the opposite sense i.e. if the hoop rotates about the pivot in a counter-clockwise direction by a given angle, the bead rotates clockwise about the centre of the hoop by the same angle. The bead remains at the lowest part of the hoop. The ratio of the masses m/M changes the frequency of this oscillation, but not the amplitude.



Any motion is a combination of these two modes.

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* The answer in the book gives $\omega'^{\,2} = \dfrac{M+m}{m}\dfrac{g}{a}$