Problem

A particle of mass m and charge q is moving in the equatorial plane z=0 of a magnetic dipole of moment μ, described by a vector potential with the single non-zero component $A_\varphi = \mu_0 \mu sin\theta/4\pi r^2$. Show that it will continue to move in the plane. Initially, it is approaching from a great distance with velocity v and impact parameter b, whose sign is defined to be that of p_φ. Show that v and p_φ are constants of the motion, and that the distance of closest approach to the dipole is $\frac{1}{2}(\sqrt{b^2 \mp a^2} \pm b)$, according as $b > a$ or $b < a$, where $a^2 = \mu_0 q\mu/\pi mv$. (Here $q\mu$ is assumed positive.) Find also the range of values of b for which the velocity can become purely radial, and the distances at which it does so. Describe qualitatively the appearance of the orbits for different values of b.

Solution

The corresponding magnetic field to the vector potential $A_\varphi = \mu_0 \mu sin\theta/4\pi r^2$ is calculated from

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\
&= \left\{ \frac{1}{rsin\theta} \left(\frac{\partial (sin\theta A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} \right), \frac{1}{rsin\theta} \frac{\partial A_r}{\partial \varphi} - \frac{1}{r} \frac{\partial (rA_\varphi)}{\partial r}, \frac{1}{r} \left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \right\} \\
&= \left\{ \frac{\mu \mu_0 cos\theta}{2\pi r^3}, \frac{\mu \mu_0 sin\theta}{4\pi r^3}, 0 \right\}
\end{aligned}$$

Since the particle is moving in the equatorial plane; $\theta = \pi/2 \quad \rightarrow \quad B_r = 0$, $B_\theta = \mu\mu_0/4\pi r^3$, $B_\varphi = 0$

B is perpendicular to z=0, so the particle will continue to move in the plane. $A_\varphi = \mu_0 \mu/4\pi\rho^2$ and; since B does no work the total energy $E = \frac{1}{2}mv^2$ remains the same. i.e. v is a constant of motion.

Lagrangian for charged particle in an electromagnetic field

$$\begin{aligned}
L &= \tfrac{1}{2} m \dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A} - q\phi \\
&= \tfrac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 \right) + q\rho\dot{\varphi} A_\varphi - q\phi \qquad \text{in polar co-ords } (\rho, \varphi) \\
&= \tfrac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 \right) + q\rho\dot{\varphi} A_\varphi
\end{aligned}$$

Generalized momenta

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \qquad \rightarrow \qquad \dot{\rho} = p_\rho/m$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2 \dot{\varphi} + q\rho A_\varphi \qquad \rightarrow \qquad \dot{\varphi} = (p_\varphi - q\rho A_\varphi)/m\rho^2$$

Hamiltonian

$$\begin{aligned}
H &= \tfrac{1}{2} m\dot{\rho}^2 + \tfrac{1}{2} m\rho^2 \dot{\varphi}^2 + q\rho\dot{\varphi} A_\varphi \\
&= \tfrac{1}{2} m\dot{\rho}^2 + \dot{\varphi} \left(\tfrac{1}{2} m\rho^2 \dot{\varphi} + q\rho A_\varphi \right) \\
&= \tfrac{1}{2} m\dot{\rho}^2 + \dot{\varphi} \left(\tfrac{1}{2} m\rho^2 \left(\frac{p_\varphi - q\rho A_\varphi}{m\rho^2} \right) + q\rho A_\varphi \right) \\
&= \tfrac{1}{2} m\dot{\rho}^2 + \frac{\dot{\varphi}}{2} \left(p_\varphi + q\rho A_\varphi \right) \\
&= \tfrac{1}{2} m\dot{\rho}^2 + \frac{p_\varphi - q\rho A_\varphi}{2m\rho^2} \left(p_\varphi + q\rho A_\varphi \right) \\
&= \tfrac{1}{2} m\dot{\rho}^2 + \frac{p_\varphi^2 - q^2 \rho^2 A_\varphi^2}{2m\rho^2} \\
&= \tfrac{1}{2} m\dot{\rho}^2 + \frac{p_\varphi^2}{2m\rho^2} - \frac{q^2 \mu^2 \mu_0^2}{32 m \pi^2 \rho^4}
\end{aligned}$$

Hamilton's equation for φ

$$-\dot{p}_\varphi = \frac{\partial H}{\partial \varphi} = 0 \quad \rightarrow \quad p_\varphi \quad \text{is a second constant of motion}$$

At great distances $\pi - \varphi = \dfrac{b}{\rho} \quad \rightarrow \quad \dot{\varphi} = \dfrac{b\dot{\rho}}{\rho^2} \quad \rightarrow \quad p_\varphi = m\rho^2\dot{\varphi} = mvb$ (neglecting magnetic field)

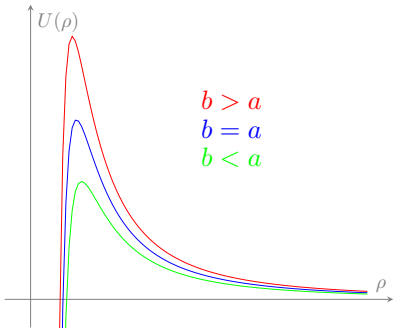
$$\begin{aligned} H &= \tfrac{1}{2}m\dot{\rho}^2 + \frac{m^2v^2b^2}{2m\rho^2} - \frac{a^4mv^2}{8\rho^4} \qquad \text{where} \quad a^2 = \mu\mu_0/4\pi mv \\ &= \tfrac{1}{2}m\dot{\rho}^2 + \tfrac{1}{2}mv^2\left(\frac{b^2}{\rho^2} - \frac{a^4}{4\rho^4}\right) \end{aligned}$$

The Hamiltonian is the total energy and at distance of closest approach $\dot{\rho} = 0$, hence

$$\begin{aligned} \tfrac{1}{2}mv^2 &= \tfrac{1}{2}mv^2\left(\frac{b^2}{\rho^2} - \frac{a^4}{4\rho^4}\right) \\ \rho^4 - \rho^2b^2 + a^4/4 &= 0 \\ \rho^2 &= \tfrac{1}{2}(b^2 \pm \sqrt{b^4 - a^4}) \end{aligned}$$

Writing the Hamiltonian in the form $H = \frac{1}{2}m\dot{\rho}^2 + U(\rho)$

$$\begin{aligned} U &= \tfrac{1}{2}mv^2\left(\frac{b^2}{\rho^2} - \frac{a^4}{4\rho^4}\right) \\ \frac{\mathrm{d}U}{\mathrm{d}\rho} &= \tfrac{1}{2}mv^2\left(-\frac{2b^2}{\rho^3} + \frac{a^4}{\rho^5}\right) \end{aligned}$$



For $U_{max} \quad \partial U/\partial\rho = 0$

$$\begin{aligned} \rho^2 &= a^4/2b^2 \\ U_{max} &= \tfrac{1}{2}mv^2\left(\frac{b^2}{\rho^2} - \frac{a^4}{4\rho^4}\right) \\ &= \tfrac{1}{2}mv^2\left(\frac{2b^4}{a^4} - \frac{b^4}{a^4}\right) \\ &= \tfrac{1}{2}mv^2\left(\frac{b}{a}\right)^4 \end{aligned}$$

For the particle to overcome the potential barrier $E > U_{max} \quad$ or $\quad b < a$

For values of $b > a$ the particle will 'bounce off' the dipole, it's trajectory being a hyperbola.

For $b = a$ the particle will follow an approximately hyperbolic path until $\dot{\rho} = 0$ and then circle the dipole with angular velocity v/ρ where $\rho^2 = \frac{1}{2}b^2$.

For $0 < b < a$ the particle will follow a similar trajectory at first, but then be 'captured' by the dipole i.e fall into the potential well. This will happen around $U = 0$ or $b^2/\rho^2 = a^4/4\rho^4$ i.e. $\rho = a^2/2b$.

Note: The answers in the book are different. Either I've got the physics wrong or there are misprints in the book. I've also used $a^2 = \mu\mu_0/4\pi mv$. Suggestions for improvement/correction welcome!