Problem

The magnetic field of a particle accelerator is axially symmetric, and in the plane z=0 has only one component $A_\varphi(\rho, z)$. A particle of mass m and charge q is moving in a circle, radius a with an angular velocity equal to the cyclotron frequency for the field at that radius. By defining $J = p_\varphi - q\rho A_\varphi$ and treating the third term in the Hamiltonian as an effective potential energy function $U(\rho, z) = J^2/2m\rho^2$, investigate the stability of the motion. Evaluate the second derivatives of U at $\rho = a$, z=0, and show that they may be written

$$\frac{\partial^2 U}{\partial \rho^2} = \frac{q^2}{m}\left[B_z\left(B_z + \rho\frac{\partial B_z}{\partial \rho}\right)\right]_{\rho=a,z=0}$$

$$\frac{\partial^2 U}{\partial \rho \partial z} = 0 \qquad\qquad \frac{\partial^2 U}{\partial z^2} = -\frac{q^2}{m}\left[B_z\rho\frac{\partial B_z}{\partial \rho}\right]_{\rho=a,z=0}$$

Given that the dependence of B_z on ρ near the equilibrium orbit is described by $B_z \propto (a/\rho)^n$, show that the orbit is stable if $0 < n < 1$.

Solution

Working in cylindrical polars throughout and using these results from Problem 14:

In the plane z=0

$$B_\rho = -\frac{\partial A_\varphi}{\partial z} \qquad B_\varphi = 0 \quad \rightarrow \quad \frac{\partial A_\rho}{\partial z} = \frac{\partial A_z}{\partial \rho} \qquad B_z = \frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial \rho}$$

Hamiltonian

$$H = \frac{p_\rho^2 + p_z^2}{2m} + \frac{J^2}{2m\rho^2} = \frac{p_\rho^2 + p_z^2}{2m} + U(\rho, z)$$

$$J = p_\varphi - q\rho A_\varphi \qquad \partial J/\partial \rho = -q\rho B_z \qquad \partial J/\partial z = q\rho B_\rho$$

Equilibrium conditions at $z = 0 \quad \rho = a$

$$U = J^2/2m\rho^2 \qquad \partial U/\partial \rho = 0 \rightarrow \mathbf{J} = -qB_z\rho^2\,\mathbf{k} \qquad \partial U/\partial z = qJB_\rho/m\rho \rightarrow B_\rho = 0$$

Evaluating the derivatives

$$\frac{\partial U}{\partial \rho} = -\frac{J^2}{m\rho^3} - \frac{qJB_z}{m\rho}$$

$$\frac{\partial^2 U}{\partial \rho^2} = \frac{3J^2}{m\rho^4} + \frac{2qJB_z}{m\rho^2} - \frac{q}{m}\left(-\frac{JB_z}{\rho^2} - qB_z^2 + \frac{J}{\rho}\frac{\partial B_z}{\partial \rho}\right)$$

$$U_{\rho\rho} = \frac{q^2}{m}\left[B_z\left(B_z + \rho\frac{\partial B_z}{\partial \rho}\right)\right]_{\rho=a,z=0}$$

$$U_{\rho z} = \frac{\partial^2 U}{\partial \rho \partial z} = 0$$

$$\frac{\partial U}{\partial z} = \frac{qJB_\rho}{m\rho}$$

$$\frac{\partial^2 U}{\partial z^2} = -\frac{qJB_\rho}{m\rho^2} + \frac{q^2 B_\rho^2}{m} + \frac{qJ}{m\rho}\frac{\partial B_\rho}{\partial z}$$

$$U_{zz} = -\frac{q^2}{m}\left[B_z\rho\frac{\partial B_z}{\partial \rho}\right]_{\rho=a,z=0} \qquad \left(\frac{\partial B_\rho}{\partial z} = 0\,;\ \frac{\partial B_\rho}{\partial z} = \frac{\partial B_z}{\partial \rho}\right)$$

Equations of motion from the Hamiltonian

$$m\ddot{\rho} = -\frac{\partial U}{\partial \rho} \qquad\qquad m\ddot{z} = -\frac{\partial U}{\partial z}$$

To investigate stability of the orbit at $\rho = a$ the equations of motion are modified using variables $P = \rho - \rho_0 \quad Z = z - z_0$ where P and Z are small quantities ($\rho_0 = a \quad z_0 = 0$), and the Taylor series approximation used for the derivatives of the potential, hence

$$m\ddot{P} = U_{\rho\rho}(P + a) + U_{\rho z}Z$$
$$m\ddot{Z} = U_{zz}Z + U_{\rho z}(P + a)$$

Where, using $B_z \propto (a/\rho)^n$

$$\begin{aligned} U_{\rho\rho} &= \frac{q^2}{m}\left[B_z\left(B_z + \rho\frac{\partial B_z}{\partial \rho}\right)\right]_{\rho=a,z=0} \\ &\propto \frac{q^2}{m}\left[\frac{a^{2n}}{\rho^{2n}}(1-n)\right]_{\rho=a,z=0} \\ U_{\rho\rho} &\propto \frac{q^2}{m}(1-n) \end{aligned}$$

$$\begin{aligned} U_{zz} &= -\frac{q^2}{m}\left[B_z\rho\frac{\partial B_z}{\partial \rho}\right]_{\rho=a,z=0} \\ &\propto \frac{q^2 n}{m}\left[\frac{a^{2n}}{\rho^{2n}}\right]_{\rho=a,z=0} \\ U_{zz} &\propto \frac{q^2 n}{m} \end{aligned}$$

Equations of motion in new variables

$$\ddot{P} \propto -(1-n)(P+a) \qquad\qquad \ddot{Z} \propto -nZ$$

or

$$\ddot{P} + \omega_P^2(P+a) = 0 \qquad\qquad \ddot{Z} + \omega_z^2 Z = 0$$

These represent stable oscillatory motion with $\omega_P^2 \propto (1-n) \quad$ and $\quad \omega_z^2 \propto n$.

Provided that angular frequencies are real numbers, hence the condition is $0 < n < 1$.
