Problem

Show that the condition that Hamilton's equations remain unchanged under the transformation generated by G is dG/dt=0 even in the case when G has an explicit time-dependence, in addition to its dependence via $q(t)$ and $p(t)$. Proceed as follows. The first set of Hamilton's equations, ($\dot{q}_\alpha = \partial H/\partial p_\alpha$), will be unchanged provided that

$$\frac{d}{dt}(\delta q_\alpha) = \delta\left(\frac{\partial H}{\partial p_\alpha}\right)$$

Write both sides of this equation in terms of G and show that it is equivalent to the condition

$$\frac{\partial}{\partial p_\alpha}\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right) = 0$$

Thus G is independent of each p_α. Similarly, by using the other set of Hamilton's equations ($\dot{p}_\alpha = -\partial H/\partial q_\alpha$), show that it is independent of each q_α. Thus dG/dt must be a function of t alone. But since we can always add to G any function of t alone without affecting the transformation it generates, this means we can choose it so that dG/dt=0.

Solution

Comparing the two sides of the condition for Hamilton's first set of equations

$$\begin{aligned}
\frac{d}{dt}(\delta q_\alpha) &: \delta\left(\frac{\partial H}{\partial p_\alpha}\right)\\
\frac{d}{dt}\left(\frac{\partial G}{\partial p_\alpha}\right) &: \left[\frac{\partial H}{\partial p_\alpha}, G\right]\\
\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_\alpha}\right) + \sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\dot{q}_\alpha + \frac{\partial}{\partial p_\alpha}\frac{\partial G}{\partial p_\alpha}\dot{p}_\alpha\right) &: \sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha}\frac{\partial H}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)\\
\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_\alpha}\right) + \sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial G}{\partial p_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha}\frac{\partial G}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha}\right) &: \sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha}\frac{\partial H}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\right)
\end{aligned}$$

Hence the condition for equality

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_\alpha}\right) = 0 \qquad \text{or} \qquad \frac{\partial}{\partial p_\alpha}\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right) = 0$$

Similarly for the second set of equations

$$\begin{aligned}
\frac{d}{dt}(\delta p_\alpha) &: \delta\left(\frac{\partial H}{\partial q_\alpha}\right)\\
-\frac{d}{dt}\left(\frac{\partial G}{\partial q_\alpha}\right) &: \left[\frac{\partial H}{\partial q_\alpha}, G\right] = -\left[G, \frac{\partial H}{\partial q_\alpha}\right]\\
\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_\alpha}\right) + \sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial G}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\dot{p}_\alpha\right) &: \sum_{\alpha=1}^{n}\left(\frac{\partial G}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha} - \frac{\partial G}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha}\frac{\partial H}{\partial q_\alpha}\right)\\
\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_\alpha}\right) + \sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial G}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha}\frac{\partial G}{\partial q_\alpha}\frac{\partial H}{\partial q_\alpha}\right) &: \sum_{\alpha=1}^{n}\left(\frac{\partial G}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha} - \frac{\partial G}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha}\frac{\partial H}{\partial q_\alpha}\right)
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_\alpha}\right) = 0 \qquad \text{or} \qquad \frac{\partial}{\partial q_\alpha}\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right) = 0$$

Hamilton's equations remain unchanged under the transformation generated by G if dG/dt=0 even in the case when G has an explicit time-dependence.