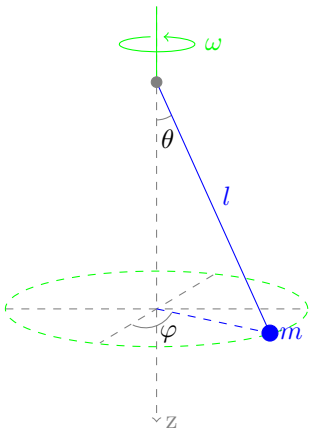
Problem

A light rigid rod, of length l, carrying a mass m at one end, is hinged at the other to a vertical axis, so that it can swing freely in a vertical plane. A torque is applied to the vertical axis and adjusted to constrain the system to rotate with constant angular velocity ω. Find the Hamiltonian function for the system, and verify that it is equal to $T' + V'$. Determine the frequency of small oscillations about the stable 'equilibrium' position when $\omega^2 > g/l$.

Solution

Using generalized co-ordinates θ and φ, the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \left(\tfrac{1}{2}ml^2\dot{\theta} + \tfrac{1}{2}ml^2\dot{\varphi}^2 sin^2\theta\right) - \left(mgl(1 - cos\theta)\right) \end{aligned}$$



This is a forced system with $\dot{\varphi} = \omega$(constant); hence

$$\begin{aligned} L &= \left(\tfrac{1}{2}ml^2\dot{\theta}^2\right) - \left(mgl(1 - cos\theta) - \tfrac{1}{2}ml^2\omega^2 sin^2\theta\right) \\ &= T' - V' \end{aligned}$$

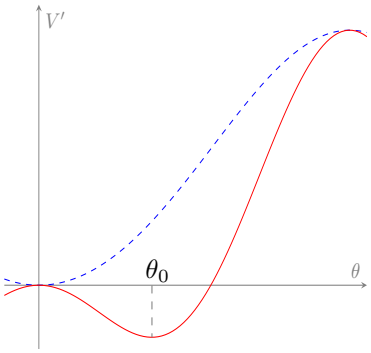
The generalised momenta p_θ and p_φ are given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}^2 \qquad\qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = 0$$

Hence the Hamiltonian

$$\begin{aligned} H(\theta, \varphi) &= (p_\theta\dot{\theta} + p_\varphi\dot{\varphi}) - L \\ &= ml^2\dot{\theta}^2 - \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - cos\theta) - \tfrac{1}{2}ml^2\omega^2 sin^2\theta \\ &= \left(\tfrac{1}{2}ml^2\dot{\theta}^2\right) + \left(mgl(1 - cos\theta) - \tfrac{1}{2}ml^2\omega^2 sin^2\theta\right) \\ H &= T' + V' \end{aligned}$$

The 'effective potential energy' curve V' for $\theta = 0$ to π, depends on the value of ω.



If $\omega^2 < g/l\,(\omega_0^2)$ [blue] there is a stable equilibrium point at θ=0.

If $\omega^2 > g/l$ [red] this equilibrium becomes unstable, though it *is* stable at the minimum point θ_0.

Hence (for $\omega^2 > g/l$) if

$$\begin{aligned} dV'/d\theta &= 0 \\ mgl sin\theta - ml^2\omega^2 sin\theta cos\theta &= 0 \\ cos\theta_0 &= g/l\omega^2 \end{aligned}$$

For small oscillations about θ_0, V' may be approximated by the Taylor series

$$V' = \tfrac{1}{2}k\theta^2 \quad \text{where} \quad k = d^2V'(\theta_0)/d\theta^2 \quad \text{hence}$$

$$\frac{d^2V'}{d\theta^2} = mglcos\theta - ml^2\omega^2 cos2\theta$$

$$k = mglcos\theta_0 - ml^2\omega^2(2cos^2\theta_0 - 1)$$

$$= mg^2/\omega^2 - 2mg^2/\omega^2 + ml^2\omega^2$$

$$k = ml^2\omega^2 - mg^2/\omega^2$$

The oscillations are simple harmonic

$$\ddot{\theta} + \Omega^2\theta = 0 \quad \text{where} \quad \Omega^2 = k/ml^2$$

$$\Omega^2 = \omega^2 - g^2/\omega^2 l^2$$
