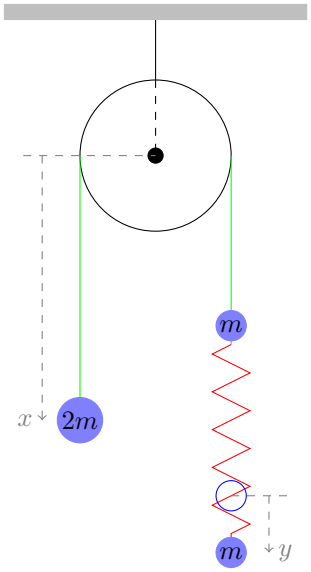
Problem

A light, inextensible string passes over a small pulley and carries a mass $2m$ on one end. On the other end is a mass m, and beneath it, supported by a spring with spring constant k, a second mass m. Find the Hamiltonian function, using the distance x of the first mass beneath the pulley, and the extension y in the spring, as generalized co-ordinates. Show that x is ignorable. To what symmetry property does this correspond? If the system is released from rest with the spring unextended, find the positions of the particles at any later time.

Solution

Lagrangian in generalized co-ordinates x & y



$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}(2m)\dot{x}^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m(\dot{x}+\dot{y})^2 - \left(\tfrac{1}{2}ky^2 - mgy\right) \\ &= \tfrac{3}{2}m\dot{x}^2 + \tfrac{1}{2}m(\dot{x}+\dot{y})^2 - \tfrac{1}{2}ky^2 + mgy \end{aligned}$$

The generalized momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = 4m\dot{x} + m\dot{y} \qquad (1)$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \; m\dot{x} + m\dot{y} \qquad (2)$$

From (1) & (2) $\dot{x} = (p_x - p_y)/3m$ and $\dot{x} + \dot{y} = p_y/m$

Hence the Hamiltonian

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \quad (\text{or } T+V) \\ &= \tfrac{3}{2}m\frac{(p_x-p_y)^2}{9m^2} + \tfrac{1}{2}m\frac{p_y^2}{m^2} + \tfrac{1}{2}ky^2 - mgy \\ &= \frac{(p_x-p_y)^2}{6m} + \frac{p_y^2}{2m} + \tfrac{1}{2}ky^2 - mgy \end{aligned}$$

Hamilton's equation for x

$$-\dot{p}_x = \frac{\partial H}{\partial x} = 0 \qquad p_x \text{ is constant} \qquad \rightarrow \qquad x \text{ is ignorable}$$

$$\begin{aligned} 4\dot{x} + \dot{y} &= c_1 \\ \dot{x} &= -\dot{y}/4 \\ x &= c_2 - y/4 \\ x &= x_0 - y/4 \end{aligned}$$

Initial conditions($t = 0$): $\dot{x} = \dot{y} = 0 \rightarrow c_1 = 0$ & $x = x_0, y = 0 \rightarrow c_2 = x_0$

If $x \rightarrow x + \delta x$ the total energy of the system is unchanged, since x does not appear explicitly in the Hamiltonian. This corresponds to a spatial translation.

Hamilton's equation for y

$$\begin{aligned}
-\dot{p}_y &= \frac{\partial H}{\partial y} = ky - mg \\
-m(\ddot{x} + \ddot{y}) &= ky - mg \\
-m(-\ddot{y}/4 + \ddot{y}) &= ky - mg \\
\ddot{y} + \frac{4k}{3m}\left(y - \frac{mg}{k}\right) &= 0 \\
y - \frac{mg}{k} &= A(cos\omega t - \varphi) \qquad \text{where} \qquad \omega^2 = \frac{4k}{3m} \\
\dot{y} &= -A\omega sin(\omega t - \varphi) \\
y &= \frac{mg}{k}\left(1 - cos\omega t\right)
\end{aligned}$$

Initial conditions: $\dot{y} = 0 \rightarrow \varphi = 0 \quad \& \quad y = 0 \rightarrow A = -\frac{mg}{k}$
