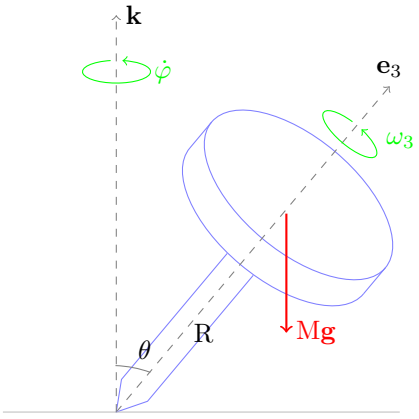
Problem

Obtain the Hamiltonian function for a symmetric top, whose pivot is free to slide on a smooth horizontal table, in terms of the generalized co-ordinates $X, Y, \varphi, \theta, \psi$, and the principal moments I_1^*, I_2^*, I_3^*. Find whether the minimum angular velocity required for stable vertical motion is greater or less than in the case of a fixed pivot. Can you explain this result physically?

Solution

Using the generalized co-ordinates (X, Y, Z) for the centre of mass of the top and Euler angles (φ, θ, ψ) for it's orientation in space, the Lagrangian is given by



$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}M\dot{Y}^2 + \tfrac{1}{2}M\dot{Z}^2 + \tfrac{1}{2}I_1^*\omega_1^2 + \tfrac{1}{2}I_2^*\omega_2^2 + \tfrac{1}{2}I_3^*\omega_3^2 - MgZ$$

Where

$$Z = Rcos\theta \qquad \dot{Z} = -Rsin\theta\,\dot{\theta}$$

$$\boldsymbol{\omega} = -\dot{\varphi}sin\theta\mathbf{e'}_1 + \dot{\theta}\mathbf{e'}_2 + (\dot{\psi} + \dot{\varphi}cos\theta)\mathbf{e}_3$$

Hence

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}M\dot{Y}^2 + \tfrac{1}{2}MR^2sin^2\theta\,\dot{\theta}^2 + \tfrac{1}{2}I_1^*(\dot{\varphi}sin\theta)^2 + \tfrac{1}{2}I_1^*\dot{\theta}^2 + \tfrac{1}{2}I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)^2 - MgRcos\theta$$

The generalized momenta from Lagrange's equations

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \quad \rightarrow \quad \dot{X} = p_X/M \tag{1}$$

$$p_Y = \frac{\partial L}{\partial \dot{Y}} = M\dot{Y} \quad \rightarrow \quad \dot{Y} = p_Y/M \tag{2}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = I_1^*\dot{\varphi}sin\theta + I_3^*(\dot{\psi} + \dot{\varphi}cos\theta)cos\theta \tag{3}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = (I_1^* + MR^2sin^2\theta)\dot{\theta} \quad \rightarrow \quad \dot{\theta} = p_\theta/(I_1^* + MR^2sin^2\theta) \tag{4}$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3^*(\dot{\psi} + \dot{\varphi}cos\theta) \tag{5}$$

From (3) & (5) $\quad \dot{\varphi} = \dfrac{p_\varphi - p_\psi cos\theta}{I_1^* sin^2\theta} \qquad \& \qquad \dot{\psi} = \dfrac{p_\psi}{I_3^*} - \dfrac{(p_\varphi - p_\psi cos\theta)cos\theta}{I_1^* sin^2\theta}$

The Hamiltonian is

$$\begin{aligned} H &= T + V \\ &= \big(p_X\dot{X} + p_Y\dot{Y} + p_\varphi\dot{\varphi} + p_\theta\dot{\theta} + p_\psi\dot{\psi}\big)/2 + MgRcos\theta \\ &= \frac{p_X^2 + p_Y^2}{2M} + \frac{p_\varphi^2 - p_\varphi p_\psi cos\theta}{2I_1^* sin^2\theta} + \frac{p_\theta^2}{2(I_1^* + MR^2sin^2\theta)} + \frac{p_\psi^2}{2I_3^*} - \frac{p_\psi(p_\varphi - p_\psi cos\theta)cos\theta}{2I_1^* sin^2\theta} + MgRcos\theta \\ &= \frac{p_X^2 + p_Y^2}{2M} + \frac{(p_\varphi - p_\psi cos\theta)^2}{2I_1^* sin^2\theta} + \frac{p_\theta^2}{2(I_1^* + MR^2sin^2\theta)} + \frac{p_\psi^2}{2I_3^*} + MgRcos\theta \end{aligned}$$

From from Hamilton's second equation ($-p_\alpha = \partial H/\partial q_\alpha$); θ is not ignorable, but φ and ψ are i.e. p_φ and p_ψ are constants of motion.

Hence

$$H = \frac{p_X^2 + p_Y^2}{2M} + \frac{p_\theta^2}{2(I_1^* + MR^2 sin^2\theta)} + U(\theta)$$

Where the effective potential energy function $U(\theta) = \dfrac{(p_\varphi - p_\psi cos\theta)^2}{2I_1^* sin^2\theta} + \dfrac{p_\psi^2}{2I_3^*} + MgRcos\theta$

$U(\theta)$ must be finite, if a top is set spinning vertically ($\theta \approx 0$) then p_φ must equal p_ψ; so setting $p_\varphi = p_\psi = I_3\omega_3$ (note $I_3^* = I_3$)

$$\begin{aligned} U(\theta) &= \frac{I_3^2\omega_3^2}{2I_1^*}\left(\frac{1-cos\theta}{sin\theta}\right)^2 + \tfrac{1}{2}I_3\omega_3^2 + MgRcos\theta \\ &= \frac{I_3^2\omega_3^2}{2I_1^*}tan^2\tfrac{1}{2}\theta + \tfrac{1}{2}I_3\omega_3^2 + MgRcos\theta \\ &\approx \left(\tfrac{1}{2}I_3^*\omega_3^2 + MgR\right) + \frac{1}{2}\left(\frac{I_3^2\omega_3^2}{4I_1^*} - MgR\right)\theta^2 \qquad (\theta \quad \text{small}) \end{aligned}$$

$U(\theta)$ has a minimum at $\theta = 0$, which is a position of stable equilibrium if the coefficient of θ^2 is positive, hence the minimum value for stability is given by

$$\omega_3^2 = \frac{4I_1^* MgR}{I_3^2}$$

This is identical to the case of a fixed pivot except that I_1 is replaced by I_1^*. Since $I_1 = I_1^* + MR^2$ the minimum stability value of ω_3^2 for the sliding pivot is reduced by a factor of I_1^*/I_1.

For values of ω greater than the minimum required for stable vertical motion it's easier for the top with the sliding pivot to right itself than for the fixed pivot. In the sliding case the pivot simply moves across the table (effectively rotating the top about its centre of mass), whereas in the fixed case the whole top has to rotate about a point some distance away; and $I_3^* < I_3$.
