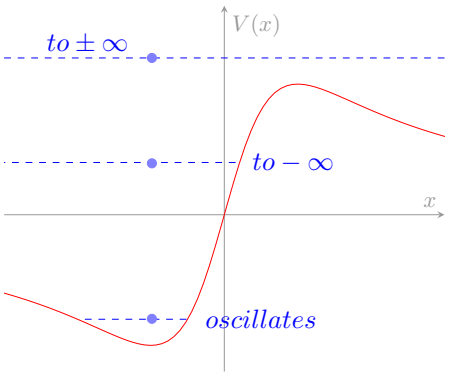
Problem

The potential energy function of a particle of mass m is $V = cx/(x^2 + a^2)$, where c and a are positive constants. Sketch V as a function of x. Find the position of stable equilibrium, and the period of small oscillations about it. Given that the particle starts from this point with velocity v, find the range of values of v for which it (a) oscillates, (b) escapes to $-\infty$, and (c) escapes to $+\infty$.

Solution

$$V = \frac{cx}{(x^2 + a^2)}$$
$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{c(x^2 + a^2) - 2cx^2}{(x^2 + a^2)^2}$$
$$= \frac{c(a^2 - x^2)}{(x^2 + a^2)^2}$$



At the turning points $dV/dx = 0$ i.e. $x = \pm a$

Equilibrium points are $(\mp a, \mp c/2a)$

Stable equilibrium is at the minimum $(-a, -c/2a)$

Near the minimum $V = V(-a) + V'(-a)\ (x + a) + V''(-a)\ (x + a)^2/2! + \ldots$ (Taylor series).

$$V'' = \frac{(x^2 + a^2)^2(-2cx) - c(a^2 - x^2)(4x(x^2 + a^2))}{(x^2 + a^2)^4}$$
$$V''(-a) = \frac{(2a^2)^2(2ca)}{(2a^2)^4} = \frac{c}{2a^3}$$

$$V \approx -\frac{c}{2a} + \frac{c}{4a^3}(x + a)^2$$
$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{c}{2a^3}(x + a)$$

Hence the equation of motion for small oscillations; and the period of oscillation

$$m\ddot{x} + \frac{c}{2a^3}(x + a) = 0$$
$$\omega = \sqrt{\frac{c}{2ma^3}}$$
$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2ma^3}{c}}$$

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a) Particle oscillates if total energy $E = V(x) + \frac{1}{2}mv^2 < 0$

$$\tfrac{1}{2}mv^2 < c/2a \qquad |v| < \sqrt{c/ma}$$

b) Particle escapes to $-\infty$

i. $E > 0$ and v is -ve

$$\tfrac{1}{2}mv^2 > c/2a \qquad\qquad v < -\sqrt{c/ma}$$

ii. $c/2a > E > 0$ and v is +ve (reflects off potential barrier)

$$c/a > \tfrac{1}{2}mv^2 > c/2a \qquad\qquad \sqrt{2c/ma} > v > \sqrt{c/ma}$$

c) Particle escapes to $+\infty$ if $E > c/2a$ and v is +ve

$$\tfrac{1}{2}mv^2 > c/a \qquad\qquad v > \sqrt{2c/ma}$$
