Problem

a) A particle falling under gravity is subject to a retarding force proportional to its velocity. Find its position as a function of time, if it starts from rest, and show that it will eventually reach a terminal velocity.

b) If the terminal speed of the particle is 50 ms^{-1}, find the time it takes to reach 40 ms^{-1}, and the distance fallen in that time.

Solution

a) The equation of motion for the particle is $m\ddot{z} = -\lambda\dot{z} - mg$, hence

$$\begin{aligned} \ddot{z} + \gamma\dot{z} &= -g && (\gamma = \lambda/m) \\ \dot{z} + \gamma z &= -gt + c \\ \dot{z} + \gamma z &= -gt && (t = 0: \quad \dot{z} = z = 0 \to c = 0) \end{aligned}$$

$$\begin{aligned} \dot{z}e^{\gamma t} + \gamma z e^{\gamma t} &= -gte^{\gamma t} \\ \frac{d}{dt}\left(ze^{\gamma t}\right) &= -gte^{\gamma t} \\ ze^{\gamma t} &= -g\int te^{\gamma t} \\ ze^{\gamma t} &= -g\left(\frac{te^{\gamma t}}{\gamma} - \frac{e^{\gamma t}}{\gamma^2} + c\right) \\ ze^{\gamma t} &= -g\left(\frac{te^{\gamma t}}{\gamma} - \frac{e^{\gamma t}}{\gamma^2} + \frac{1}{\gamma^2}\right) \quad (t = 0: \quad z = 0 \to c = 1/\gamma^2) \end{aligned}$$

$$\begin{aligned} z &= -\frac{gt}{\gamma} + \frac{g}{\gamma^2}(1 - e^{-\gamma t}) && (1) \\ \dot{z} &= -\frac{g}{\gamma} + \frac{g\gamma e^{-\gamma t}}{\gamma^2} \\ \dot{z} &= \frac{g}{\gamma}(e^{-\gamma t} - 1) && (2) \end{aligned}$$

As $t \to \infty \quad \dot{z} \to -g/\gamma$ i.e. terminal speed.

b) Terminal speed of $50\,ms^{-1} \to \gamma = g/50 = 1/5$.

Time taken to reach 40 ms^{-1} from (2)

$$\begin{aligned} -40 &= 50(e^{-t/5} - 1) \\ e^{t/5} &= 1/0.2 = 5 \\ t &= 5\,ln5 = 5 \times 1.61 = 8.05\,s \end{aligned}$$

Distance fallen from (1)

$$\begin{aligned} z &= -50(8.05) + 250(1 - e^{-8.05/5}) \\ &= -402.5 + 200.0 = -202.5\,m \end{aligned}$$

To reach 40 ms^{-1} takes 8.05 s during which the particle falls 202.5 m.