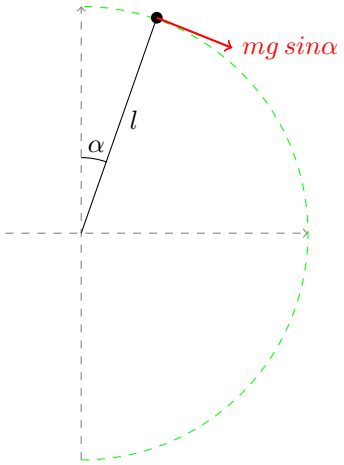
Problem

A simple pendulum comprises a bob of mass m supported by a light rigid rod of length l. Find the equation of motion for small displacements from the postion of *un*stable equilibrium, $\theta = \pi$. Show that if it is released from rest at a small angle to the upward vertical, then the time taken for the angular displacement to increase by a factor of 10 will be approximately $\sqrt{l/g}\, ln\, 20$. Evaluate this time for a pendulum of period $2s$, and find the angular velocity of the pendulum when it reaches the downward vertical.

Solution

Equation of motion is



$$\begin{aligned} ml\ddot{\alpha} &= mgsin\alpha \\ \ddot{\alpha} &= \frac{g}{l}\,\alpha \qquad\qquad \text{since } \alpha \text{ is small} \\ \ddot{\alpha} - \frac{g}{l}\,\alpha &= 0 \end{aligned}$$

The solution is

$$\begin{aligned} \alpha &= \frac{1}{2}Ae^{pt} + \frac{1}{2}Be^{-pt} \qquad (p = \sqrt{g/l}) \\ \dot{\alpha} &= \frac{1}{2}Ape^{pt} - \frac{1}{2}Bpe^{-pt} \end{aligned}$$

Initial conditions: $t = 0 \quad \dot{\alpha} = 0 \quad \alpha = \alpha_0 \quad \Rightarrow \quad A = B = \alpha_0$ hence

$$\begin{aligned} \alpha &= \frac{1}{2}\alpha_0(e^{pt} + e^{-pt}) \\ &= \alpha_0\, cosh\, pt \\ t &= \sqrt{\frac{l}{g}}\, arccosh\frac{\alpha}{\alpha_0} \\ &= \sqrt{\frac{l}{g}}\, ln\left(\frac{\alpha}{\alpha_0} + \sqrt{\left(\frac{\alpha}{\alpha_0}\right)^2 - 1}\right) \\ t_1 &= \sqrt{\frac{l}{g}}\, ln\left(10 + \sqrt{10^2 - 1}\right) \\ &\approx \sqrt{\frac{l}{g}}\, ln\, 20 \end{aligned}$$

Where t_1 is the time taken for $\alpha \to 10\,\alpha_0$

$$\begin{aligned} \sqrt{\frac{l}{g}} &= \frac{\tau}{2\pi} = \frac{1}{\pi} \\ t_1 &= ln\, 20/\pi = .954 \end{aligned}$$

Time taken for angular displacement to increase by factor of 10 is $.95\, s^{-1}$

Conservation of energy

$$\begin{aligned} \frac{1}{2}mv^2 &= 2lmg \quad \text{where} \quad l = g/\pi^2 \\ v^2 &= (l\,\dot{\alpha})^2 = 4lg \\ \dot{\alpha}^2 &= \frac{4g}{l} \quad = 4\pi^2 \\ \dot{\alpha} &= 2\pi \end{aligned}$$

Angular velocity of the pendulum on reaching downward vertical is $2\pi\ rads\,s^{-1}$.
