Problem

Solve the problem of an oscillator under a periodic force $F = F_1 cos\omega_1 t$ (turned on at $t = 0$) by the Green's function method, and verify the solution

$$x = a_1 cos(w_1 t - \theta_1) + a e^{-\gamma t} cos(\omega t - \theta).$$

Assume that the damping is less than critical. To do the integral write $sin\omega t$ as $(e^{i\omega t} - e^{-i\omega t})/2i$.

Solution

For an unforced oscillator with small damping ($\gamma < \omega_0$) initially at rest and given an impulse I

$$x(t) = \begin{cases} 0 & t < 0 \\ \dfrac{I}{m\omega}\, e^{-\gamma t} sin\omega t & t > 0 \end{cases}$$

Hence Green's function for the oscillator is

$$G(t-t') = \begin{cases} 0 & t < 0 \\ \dfrac{1}{m\omega}\, e^{-\gamma(t-t')} sin\omega(t-t') & t > 0 \end{cases}$$

Then under a simple periodic force $F = F_1 cos\omega_1 t$

$$x(t) = \int_0^t G(t-t')\, F(t')\, dt' \quad (+ \text{ transient})$$

$$= \int_0^t \frac{1}{m\omega}\, e^{-\gamma(t-t')} sin\omega(t-t')\, F_1 cos\omega_1 t'\, dt'$$

$$= \frac{F_1}{2m\omega} \int_0^t e^{-\gamma(t-t')} \Big(sin(\omega(t-t') + \omega_1 t') + sin(\omega(t-t') - \omega_1 t') \Big)\, dt'$$

$$= \frac{F_1}{4im\omega} \int_0^t e^{-\gamma(t-t')} \Big(e^{i[\omega(t-t')+\omega_1 t']} - e^{-i[\omega(t-t')+\omega_1 t']} + e^{i[\omega(t-t')-\omega_1 t']} - e^{-i[\omega(t-t')-\omega_1 t']} \Big)\, dt'$$

$$= \frac{F_1}{4im\omega} \int_0^t \Big(e^{(-\gamma+i\omega)(t-t')+i\omega_1 t'} - e^{(-\gamma-i\omega)(t-t')-i\omega_1 t'} + e^{(-\gamma+i\omega)(t-t')-i\omega_1 t'} - e^{(-\gamma-i\omega)(t-t')+i\omega_1 t'} \Big)\, dt'$$

$$= \frac{F_1}{4im\omega} \left[\frac{e^{(-\gamma+i\omega)(t-t')+i\omega_1 t'}}{\gamma + i(\omega_1 - \omega)} - \frac{e^{(-\gamma-i\omega)(t-t')-i\omega_1 t'}}{\gamma - i(\omega_1 - \omega)} + \frac{e^{(-\gamma+i\omega)(t-t')-i\omega_1 t'}}{\gamma - i(\omega_1 + \omega)} - \frac{e^{(-\gamma-i\omega)(t-t')+i\omega_1 t'}}{\gamma + i(\omega_1 + \omega)} \right]_0^t$$

$$= \frac{F_1}{4im\omega} \left[\frac{e^{i\omega_1 t} - e^{(-\gamma+i\omega)t}}{\gamma + i(\omega_1 - \omega)} - \frac{e^{-i\omega_1 t} - e^{(-\gamma-i\omega)t}}{\gamma - i(\omega_1 - \omega)} + \frac{e^{-i\omega_1 t} - e^{(-\gamma+i\omega)t}}{\gamma - i(\omega_1 + \omega)} - \frac{e^{i\omega_1 t} - e^{(-\gamma-i\omega)t}}{\gamma + i(\omega_1 + \omega)} \right]$$

$$= \frac{F_1}{4im\omega} \left[\frac{-2i(\omega_1 - \omega) cos\omega_1 t + 2i\gamma sin\omega_1 t + \text{transient}}{\gamma^2 + (\omega_1 - \omega)^2} + \frac{2i(\omega_1 + \omega) cos\omega_1 t - 2i\gamma sin\omega_1 t + \text{transient}}{\gamma^2 + (\omega_1 + \omega)^2} \right]$$

Hence, neglecting transient terms in $e^{-\gamma t}$, we have

$$x(t) = \frac{F_1}{2m\omega}\left[\frac{-(\omega_1-\omega)cos\omega_1 t + \gamma sin\omega_1 t}{\gamma^2+(\omega_1-\omega)^2} + \frac{(\omega_1+\omega)cos\omega_1 t - \gamma sin\omega_1 t}{\gamma^2+(\omega_1+\omega)^2}\right]$$

$$= \frac{F_1}{2m\omega}\left[\left(\frac{\omega_1+\omega}{\gamma^2+(\omega_1+\omega)^2} - \frac{\omega_1-\omega}{\gamma^2+(\omega_1-\omega)^2}\right) cos\omega_1 t + \left(\frac{\gamma}{\gamma^2+(\omega_1-\omega)^2} - \frac{\gamma}{\gamma^2+(\omega_1+\omega)^2}\right) sin\omega_1 t\right]$$

Since $\omega_0^2 = \gamma^2 + \omega^2$

$$x(t) = \frac{F_1}{2m\omega}\left[\left(\frac{\omega_1+\omega}{\omega_0^2+\omega_1^2+2\omega_1\omega} - \frac{\omega_1-\omega}{\omega_0^2+\omega_1^2-2\omega_1\omega}\right) cos\omega_1 t + \left(\frac{\gamma}{\omega_0^2+\omega_1^2-2\omega_1\omega} - \frac{\gamma}{\omega_0^2+\omega_1^2+2\omega_1\omega}\right) sin\omega_1 t\right]$$

$$= \frac{F_1}{2m\omega}\left[\left(\frac{2\omega(\omega_0^2+\omega_1^2) - 4\omega\omega_1^2}{(\omega_0^2+\omega_1^2)^2 - 4\omega_1^2\omega^2}\right) cos\omega_1 t + \left(\frac{4\gamma\omega\omega_1}{(\omega_0^2+\omega_1^2)^2 - 4\omega_1^2\omega^2}\right) sin\omega_1 t\right]$$

$$= \frac{F_1}{m}\left[\left(\frac{\omega_0^2-\omega_1^2}{(\omega_0^2-\omega_1^2)^2 + 4\omega_1^2\gamma^2}\right) cos\omega_1 t + \left(\frac{2\gamma\omega_1}{(\omega_0^2-\omega_1^2)^2 - 4\omega_1^2\gamma^2}\right) sin\omega_1 t\right]$$

$$= a_1\, cos(\omega_1 t - \theta_1)$$

Where

$$a_1\, cos\theta_1 = \frac{F_1}{m}\left(\frac{\omega_0^2-\omega_1^2}{(\omega_0^2-\omega_1^2)^2+4\omega_1^2\gamma^2}\right) \qquad \text{and} \qquad a_1\, sin\theta_1 = \frac{F_1}{m}\left(\frac{2\gamma\omega_1}{(\omega_0^2-\omega_1^2)^2+4\omega_1^2\gamma^2}\right)$$

Squaring and adding these equations

$$a_1^2 = \frac{F_1^2}{m^2}\left(\frac{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}{((\omega_0^2-\omega_1^2)^2+4\omega_1^2\gamma^2)^2}\right)$$

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\omega_1^2\gamma^2}}$$

Dividing

$$tan\theta_1 = \frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}$$

The general solution is then given by adding the solution of the corresponding homogeneous equation, hence

$$x(t) = a_1\, cos(\omega_1 t - \theta_1) + ae^{-\gamma t} cos(\omega t - \theta)$$

Where a and θ are constants fixed by the initial conditions.
