Problem

a) A particle moving under a conservative force oscillates between x_1 and x_2. Show that the period of oscillation is

$$\tau = 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2[V(x_2)-V(x)]}}\,dx$$

In particular, if $V = \frac{1}{2}m\omega_0^2(x^2 - bx^4)$, show that the period for oscillations of amplitude a is

$$\tau = \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}\sqrt{1-b(a^2+x^2)}}$$

Using the binomial theorem to expand in powers of b, and the substitution $x = asin\theta$, show that for small amplitude the period is approximately $\tau \approx 2\pi(1 + \frac{3}{4}ba^2)/\omega_0$.

b) Use this result to obtain an estimate of the correction factor to the period of a simple pendulum when its angular amplitude is 30°.

Solution

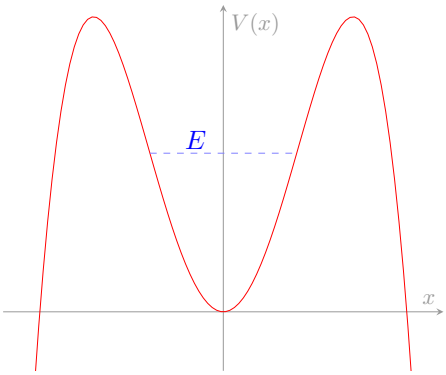
a) When $\dot{x} = 0 \quad E = V(x_1) = V(x_2)$, hence

$$\tfrac{1}{2}m\dot{x}^2 + V(x) = V(x_1)$$

$$\dot{x} = \sqrt{\frac{2\big(V(x_1)-V(x)\big)}{m}}$$

$$\tau = 2\int_{x_1}^{x_2} \frac{dx}{\dot{x}}$$

$$= 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2\big(V(x_1)-V(x)\big)}}\,dx$$



$V(x) = \frac{1}{2}m\omega_0^2(x^2 - bx^4) \quad \rightarrow \quad V(x_1) = \frac{1}{2}m\omega_0^2(a^2 - ba^4)$

$$\tau = 2\int_{-a}^{a} \sqrt{\frac{m}{m\omega_0^2\big((a^2-ba^4)-(x^2-bx^4)\big)}}\,dx$$

$$= \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{(a^2-x^2)-b(a^4-x^4)}}$$

$$= \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}\sqrt{1-b(a^2+x^2)}}$$

$$\approx \frac{2}{\omega_0}\int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}}\left(1+\frac{b(a^2+x^2)}{2}\right)dx \qquad (b \text{ small})$$

Subs $x = asin\theta \quad dx = acos\theta\,d\theta \quad \theta_1 = -\pi/2 \quad \theta_2 = \pi/2$

$$\tau = \frac{2}{\omega_0}\int_{-\pi/2}^{\pi/2} \frac{1}{acos\theta}\left(1+\frac{ba^2(1+sin^2\theta)}{2}\right)(acos\theta\,d\theta)$$

$$= \frac{2}{\omega_0}\int_{-\pi/2}^{\pi/2} 1 + \frac{ba^2}{2} + \frac{ba^2}{2}\left(\frac{1-cos2\theta}{2}\right)d\theta$$

$$\tau = \frac{2}{\omega_0}\left[\theta + \frac{3ba^2}{4}\theta - \frac{ba^2}{8}sin2\theta\right]_{-\pi/2}^{\pi/2} = \frac{2\pi}{\omega_0}\left(1+\frac{3ba^2}{4}\right)$$

b) For a simple pendulum

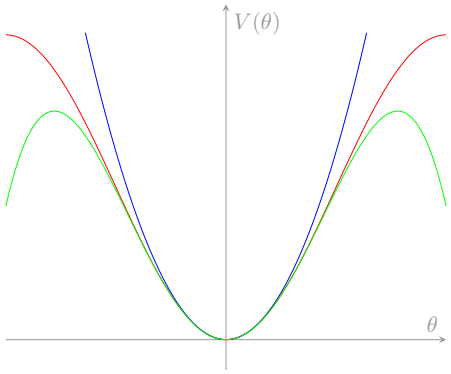
$$\begin{aligned} V(x) &= mgl(1 - cos\theta) \\ &\approx mgl\left(1 - \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right)\right) \\ &= mgl\left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right) \\ &= \tfrac{1}{2}m\omega_0^2 l^2\left(\theta^2 - \frac{\theta^4}{12}\right) \end{aligned}$$

Comparing with $V(x) = \frac{1}{2}m\omega_0^2(x^2 - bx^4) \quad \rightarrow \quad b = 1/12$

And since $\theta = \pi/6$

$$\begin{aligned} \tau &= \frac{2\pi}{\omega_0}\left(1 + \frac{3ba^2}{4}\right) \\ &= \frac{2\pi}{\omega_0}\left(1 + \frac{(\pi/6)^2}{16}\right) \\ &= \frac{2\pi}{\omega_0}(1 + \frac{(\pi/6)^2}{16}) \\ &= \frac{2\pi}{\omega_0}(1.017) \\ \tau &= 1.017 \times 2\pi\sqrt{l/g} \end{aligned}$$

Plot of the actual potential ($-\pi$ to π) versus the approximations:



$V(\theta) = mgl(1 - cos\theta)$ - actual potential
$V(\theta) = mgl\big(1 - (1 - \theta^2/2)\big)$ - small angle approximation
$V(\theta) = mgl\big(1 - (1 - \theta^2/2 + \theta^4/24)\big)$ - better approximation for small angles

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