Problem

A particle starts from rest and slides down a smooth curve under gravity. Find the shape of the curve that will minimize the time taken between two given points. Take the origin as starting point and the z-axis downwards. Show that the time taken is

$$\int_0^{z_1} \left[\frac{1+(dx/dz)^2}{2gz}\right]^{1/2} dz,$$

and hence that for a minimum

$$\left(\frac{dx}{dz}\right)^2 = \frac{c^2 z}{1-c^2 z},$$

where c is a constant of integration. To complete the integration, use the substitution $z = c^{-2} sin^2\theta$. This famous curve is known as the *brachistochrone*. It is in fact an example of a *cycloid*, the locus of a point on the rim of a circle of radius $1/2c^2$ being rolled beneath the x-axis.

Solution

Time taken (t) for the particle to slide down is given by the path integral.

$$t = \int_0^{z_1} \frac{ds}{v}$$

Where the path ds is given by $ds^2 = dx^2 + dz^2$ and the particle velocity v is given by the conservation of energy equation $\frac{1}{2}mv^2 = mgz$. Hence

$$t = \int_0^{z_1} \left[\frac{1+(dx/dz)^2}{2gz}\right]^{1/2} dz$$

Let $x' = \dfrac{dx}{dz}$ and $f(x,x') = \left(\dfrac{1+(x')^2}{2gz}\right)^{1/2}$ then

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x'} = \frac{1}{\sqrt{2g}} \frac{x'}{\sqrt{z(1+(x')^2)}}$$

Hence for a stationary value (using the Euler Lagrange equation)

$$\frac{d}{dz}\left(\frac{x'}{\sqrt{z(1+(x')^2)}}\right) = 0 \quad \text{or} \quad \frac{x'}{\sqrt{z(1+(x')^2)}} = c \qquad \text{(constant)}$$

$$(x')^2 = c^2 z(1+(x')^2)$$

$$(x')^2 = \frac{c^2 z}{1-c^2 z}$$

$$\left(\frac{dx}{dz}\right)^2 = \frac{c^2 z}{1-c^2 z}$$

Let $z = c^{-2}sin^2\theta$

$$dz = 2c^{-2}sin\theta cos\theta d\theta$$
$$dx = \sqrt{\frac{sin^2\theta}{1-sin^2\theta}} \quad \frac{2sin\theta cos\theta}{c^2}\, d\theta$$
$$x = \frac{1}{c^2}\int 2sin^2\theta\, d\theta$$
$$x = \frac{\theta - \frac{1}{2}sin2\theta}{c^2} + k$$

Initial conditions $x = 0$ when $z = 0$ and $\theta = 0 \quad \Rightarrow \quad k = 0$

Thus the solution (in parametric form) is

$$x = \frac{\theta - \frac{1}{2}sin2\theta}{c^2} \qquad\qquad z = \frac{sin^2\theta}{c^2}$$

