Problem

Find the geodesics on a sphere of unit radius. [*Hint*: Use θ as independent variable, and look for a path $\varphi = \varphi(\theta)$. To perform the integration, use the substitution $x = cot\theta$.]

Solution

Surface element on a sphere of unit radius $dA = (sin\theta d\varphi)(d\theta)$, hence an element ds of the geodesic (taking $\varphi' = d\varphi/d\theta$)

$$\begin{aligned} ds^2 &= (sin\theta d\varphi)^2 + (d\theta)^2 \\ ds &= (sin^2\theta\,\varphi'^{\,2} + 1)^{1/2} \\ s &= \int (sin^2\theta\,\varphi'^{\,2} + 1)^{1/2}\, d\theta \\ &= \int f(\varphi\varphi')\, d\theta \end{aligned}$$

Euler Lagrange

$$\begin{aligned} \frac{\partial f}{\partial \varphi} - \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial f}{\partial \varphi'}\right) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{sin^2\theta\,\varphi'}{(sin^2\theta\,\varphi'^{\,2} + 1)^{1/2}}\right) &= 0 \\ (sin^2\theta\,\varphi')^2 &= c^2(sin^2\theta\,\varphi'^{\,2} + 1) \qquad (c \text{ constant}) \\ sin^4\theta\,\varphi'^{\,2} &= c^2 sin^2\theta\,\varphi'^{\,2} + c^2 \\ \varphi'^{\,2} &= \frac{c^2}{sin^4\theta - c^2 sin^2\theta} \\ \varphi &= \int \sqrt{\frac{c^2}{sin^4\theta - c^2 sin^2\theta}}\, d\theta \end{aligned}$$

Subs
$x = cot\theta \quad \rightarrow \quad dx = -csc^2\theta d\theta = -(1 + cot^2\theta)d\theta = -(1 + x^2)d\theta \quad \rightarrow \quad d\theta = -1/(1 + x^2)\, dx$
And
$sin^2\theta = 1/csc^2\theta = 1/(1 + cot^2\theta) = 1/(1 + x^2)$

$$\begin{aligned} \varphi &= \int \sqrt{\frac{c^2}{1/(1+x^2)^2 - c^2/(1+x^2)}}\left(\frac{-dx}{1+x^2}\right) \\ &= \int \sqrt{\frac{c^2(1+x^2)^2}{1 - c^2(1+x^2)}}\left(\frac{-dx}{1+x^2}\right) \\ &= \int \frac{-c}{\sqrt{(1-c^2) - c^2x^2}}\, dx \\ &= \int \frac{-1}{\sqrt{a^2 - x^2}}\, dx \qquad \text{where} \quad a^2 = (1-c^2)/c^2 \\ \varphi &= arccos(x/a) + \varphi_0 \\ x &= acos(\varphi - \varphi_0) \\ cot\theta &= acos(\varphi - \varphi_0) \\ cot\theta &= cot\theta_0\, cos(\varphi - \varphi_0) \qquad (\varphi = \varphi_0, \theta = \theta_0 \quad \rightarrow \quad a = cot\theta_0) \end{aligned}$$