Problem

Write down the equations of motion in polar co-ordinates for a particle of unit mass moving in a plane under a force with potential energy function $V = -k\,ln(r) + cr + gr\,cos\theta$ where k, c and g are positive constants. Find the positions of equilibrium (a) if $c > g$, and (b) if $c < g$. By considering the equations of motion near these points, determine whether the equilibrium is stable (*i.e.*, will the particle, if given a small displacement, tend to return repeatedly?).

Solution

$$L = T - V = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta)$$

Lagrange equation gives

$$\ddot{r} = r\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \text{and} \qquad r^2\ddot{\theta} = -\frac{\partial V}{\partial \theta}$$

For an equilibrium point $\dfrac{\partial V}{\partial r}$ and $\dfrac{\partial V}{\partial \theta}$ must both be zero.

$$\frac{\partial V}{\partial r} = -\frac{k}{r} + c + gcos\theta = 0 \Rightarrow r = \frac{k}{c + gcos\theta}$$

$$\frac{\partial V}{\partial \theta} = -grsin\theta = 0 \Rightarrow \theta = 0 \quad \text{or} \quad \pi$$

(a) $c > g$

$$\theta = 0 \quad \text{and} \quad r = k/(c+g)$$

For small changes in r the potential increases, but for small changes in θ it decreases ie. equilibrium unstable (saddle point in V).

$$\theta = \pi \quad \text{and} \quad r = k/(c-g)$$

For small changes in both r and θ the potential increases i.e. stable equilibrium (minimum V).

(b) $c < g$

$$\theta = 0 \quad \text{and} \quad r = k/(c+g)$$

For small changes in r the potential increases, but for small changes in θ it decreases ie. equilibrium unstable (saddle point in V).

$$\theta = \pi \quad \text{and} \quad r = k/(c-g)$$

There are no equilibrium points in this region.







