Problem

The method of *Lagrange multipliers* can be extended to the calculus of variations. To find the maxima and minima of an integral I, subject to the condition that another integral $J = 0$, we have to find the stationary points on the integral $I - \lambda J$ under variations of the function $y(x)$ *and* the parameter λ. Apply this method to the *catenary*, the shape in which a uniform heavy chain hangs between two fixed supports. [The required shape is the one that minimizes the total potential energy, subject to the condition that the total length is fixed. Show that this leads to a variational problem with

$$f(y, y') = (y - \lambda)\sqrt{1 + y'^2},$$

and hence to the equation

$$(y - \lambda)\, y'' = 1 + y'^2.$$

Solve this equation by introducing the new variable $u = y'$ and solving for $u(y)$.]

Solution

For a chain of length l and mass per unit length ρ, the total length L and potential energy V are

$$\begin{aligned} L &= \int_{x_0}^{x_1} dl \qquad \text{where} \quad dl = \sqrt{1 + y'^2}\, dx \\ V &= \int_{x_0}^{x_1} \rho g y\, dl \\ &= \rho g \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx \end{aligned}$$

The problem is to stationarize $V - \lambda L$ or equivalently $f(y, y')$ where

$$\begin{aligned} f(y, y') &= y\sqrt{1 + y'^2} - \lambda\sqrt{1 + y'^2} \\ &= (y - \lambda)\sqrt{1 + y'^2} \end{aligned}$$

Euler Lagrange

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 0 \\ \sqrt{1 + y'^2} - \frac{d}{dx}\left(\frac{1}{2}(1 - y'^2)^{-1/2}(2y')(y - \lambda)\right) &= 0 \\ \frac{d}{dx}\,\frac{y'(y - \lambda)}{\sqrt{1 + y'^2}} &= \sqrt{1 + y'^2} \\ \frac{\sqrt{1 + y'^2}\left(y'^2 + (y - \lambda)y''\right) - y'(y - \lambda)(1 + y'^2)^{-1/2}(y'y'')}{(1 + y'^2)} &= \sqrt{1 + y'^2} \\ \frac{(1 + y'^2)\left(y'^2 + (y - \lambda)y''\right) - y'(y - \lambda)(y'y'')}{(1 + y'^2)^{3/2}} &= \sqrt{1 + y'^2} \\ y'^2(1 + y'^2) + y''(y - \lambda) &= (1 + y'^2)^2 \\ y''(y - \lambda) &= 1 + y'^2 \end{aligned}$$

Subs $u = \dfrac{dy}{dx}$ $\Rightarrow$ $\dfrac{d^2y}{dx^2} = \dfrac{du}{dx} = \dfrac{dy}{dx}\dfrac{du}{dy} = u\dfrac{du}{dy}$

$$u\frac{du}{dy}(y-\lambda) = 1+u^2$$

$$u^2 = \frac{1}{a^2}(y-\lambda)^2 - 1 \qquad \text{where } a \text{ is a constant}$$

$$a^2\left(\frac{dy}{dx}\right)^2 = (y-\lambda)^2 - a^2$$

$$\frac{dy}{\sqrt{(y-\lambda)^2-a^2}} = \frac{dx}{a}$$

$$arc\,cosh\left(\frac{y-\lambda}{a}\right) + \frac{b}{a} = \frac{x}{a} \qquad \text{where } b \text{ is a constant}$$

$$y = \lambda + a\,cosh\left(\frac{x-b}{a}\right)$$

Example catenary

