Problem

A star of mass M and radius R is moving with velocity v through a cloud of particles of density ρ. If all the particles that collide with the star are trapped by it, show that the mass of the star will increase at a rate

$$\frac{dM}{dt} = \pi\rho v\left(R^2 + \frac{2GMR}{v^2}\right).$$

Given that $M = 10^{31}$ kg and $R = 10^8$ km, find how the effective cross-sectional area compares with the geometric cross-section πR^2 for velocities of 1000 $km\,s^{-1}$, 100 $km\,s^{-1}$ and 10 $km\,s^{-1}$.

Solution

Consider a particle of mass m with impact parameter b at a distance r such that it's *effective potential energy* is negligible, then in the frame of reference of the star

$$\begin{aligned} E = \frac{1}{2}m\dot{r}^2 &= \frac{1}{2}mv^2 \quad \text{Total energy of particle} \\ J = m\dot{r}b &= mvb \quad \text{Angular momentum of particle (constant)} \end{aligned}$$

The maximum impact parameter for a particle to be collected is when it just touches the outer surface of the star i.e. $\dot{r} = 0$, hence applying the radial energy equation

$$\begin{aligned} E &= \frac{J^2}{2mr^2} + V(r) \\ \frac{1}{2}mv^2 &= \frac{(mvb)^2}{2mR^2} - \frac{GMm}{R} \\ b^2 &= R^2 + \frac{2GMR}{v^2} \end{aligned}$$

The mass of the star increases at a rate of $\pi\rho v b^2$ hence

$$\frac{dM}{dt} = \pi\rho v\left(R^2 + \frac{2GMR}{v^2}\right)$$

$$v = 10^6 ms^{-1} \quad \Rightarrow \quad \frac{\pi b^2}{\pi R^2} = 1 + \frac{2GM}{v^2 R} = 1 + \frac{2 \times (6.673 \times 10^{-11}) \times 10^{31}}{10^{12} \times 10^{11}} = 1.01$$

$$v = 10^5 ms^{-1} \quad \Rightarrow \quad \frac{\pi b^2}{\pi R^2} = 1 + \frac{2GM}{v^2 R} = 1 + 1.33 = 2.33$$

$$v = 10^4 ms^{-1} \quad \Rightarrow \quad \frac{\pi b^2}{\pi R^2} = 1 + \frac{2GM}{v^2 R} = 1 + 133 = 134$$

i.e. The effective cross-sectional area of the star is much higher at low velocities.
