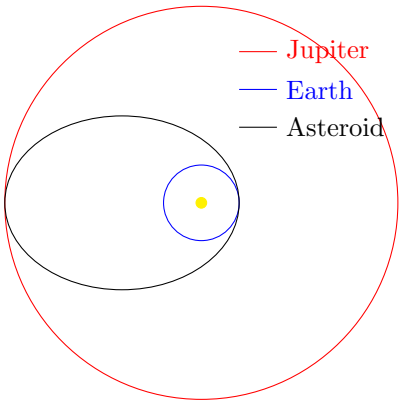
Problem

The orbit of an asteroid extends from Earth's to Jupiter's just touching both. Suppose that the asteroid approaches the Earth with an impact parameter of $5R_E$, where R_E= Earth's radius, moving in the same plane and overtaking it. Find the distance of closest approach and the angle through which the asteroid is scattered, in the frame of reference in which the Earth is at rest. (Assume that the asteroid is small enough to have negligible effect on the Earth's orbit.) What is its new velocity v relative to the Sun? Show that the semi-major axis of it's new orbit is $a_E v_E^2/(2v_E^2 - v^2)$, where a_E and v_E are the Earth's orbital radius and orbital velocity. Find the asteroid's new orbital period.

Solution

The encounter takes place at the perihelion of the asteroid's orbit, its semi-major axis a_A is $(a_J + a_E)/2 = 3.1\,AU$. For an elliptic orbit



$$a_A = \frac{|k|}{2\,|E|} \quad \text{where} \quad E \text{ is its total energy}$$

$$E = -\frac{GM_S m_A}{2a_A} = -\frac{GM_S m_A}{6.2AU}$$

Hence its speed at the perihelion is given by

$$\frac{1}{2}m_A v_A^2 - \frac{GM_S m_A}{1AU} = -\frac{GM_S m_A}{6.2AU}$$

$$v_A^2 = \frac{2GM_S}{1AU}\,\frac{5.2}{6.2}$$

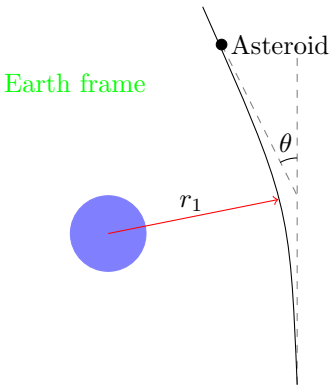
$$= \frac{2(1.327\times 10^{20})\times 5.2}{6.2(1.496\times 10^{11})}$$

$$v_A = 3.857\times 10^4\, ms^{-1}$$

For the circular Earth orbit its speed is given by $2T = V$ hence

$$v_E^2 = \frac{GM_S}{1AU} = \frac{1.327\times 10^{20}}{1.496\times 10^{11}} \quad \text{or} \quad v_E = 2.978\times 10^4\, ms^{-1}$$

In the Earth frame the asteroid is approaching at velocity $v = 8.787\times 10^3\, ms^{-1}$ with an impact parameter b of $5R_E$. The distance of closest approach r_1 is given by the equation



$$r_1 = -a + \sqrt{a^2 + b^2}$$

where $a = \dfrac{GM_E}{v^2} = \dfrac{3.986\times 10^{14}}{(8.787\times 10^3)^2} = 0.809\,R_E$ hence

$$r_1 = (-.809 + \sqrt{.809^2 + 5^2}\,)\,R_E$$

$$= 4.256\,R_E$$

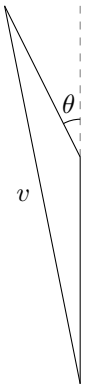
And the scattering equation gives the value of angle θ

$$b = \frac{|k|}{m_A v^2}\,cot\left(\frac{\theta}{2}\right)$$

$$cot\left(\frac{\theta}{2}\right) = \frac{m_A b v^2}{GM_E m_A} = \frac{b}{a} = \frac{5}{.809} = 6.18 \quad \Rightarrow \quad \frac{\theta}{2} = 9.19^\circ$$

The distance of closest approach is $4.26\,R_E$ and the scattering angle of the asteroid is 18.4°.

After the encounter the velocity of the asteroid v is given by



Sun frame

$$\begin{aligned} v^2 &= v_E^2 + v_A^2 + 2v_E v_A \, cos\theta \\ &= (2.978 \times 10^4)^2 + (8.787 \times 10^3)^2 \\ &\quad + 2(2.978 \times 10^4)(8.787 \times 10^3)(.9489) \\ v &= 3.822 \times 10^4 \, ms^{-1} \end{aligned}$$

Asteroid's new velocity relative to the Sun is $38.2 \, kms^{-1}$

For Earth $\dfrac{GM_S M_E}{a_E} = M_E v_E^2 \quad \Rightarrow \quad GM_S = a_E v_E^2$

For asteroid, total energy $E = -\dfrac{GM_S m_A}{2a} = -\dfrac{a_E v_E^2 m_A}{2a}$

Equating E with the values of T and V for the asteroid's current position in its new orbit

$$\begin{aligned} -\frac{a_E v_E^2 m_A}{2a} &= \frac{1}{2} m_A v^2 - \frac{GM_S m_A}{a_E} \\ \frac{a_E v_E^2}{a} &= -v^2 + 2v_E^2 \\ a &= \frac{a_E v_E^2}{2v_E^2 - v^2} \\ &= \frac{(2.978 \times 10^4)^2}{2(2.978 \times 10^4)^2 - (3.822 \times 10^4)^2} \, AU \\ &= 2.834 \, AU \end{aligned}$$

The semi-major axis of the new orbit is $\dfrac{a_E v_E^2}{2v_E^2 - v^2}$ or $2.83 AU$.

Kepler's law

$$\begin{aligned} \left(\frac{\tau}{2\pi}\right)^2 &= \frac{a^3}{GM_S} \\ &= \frac{(2.834 AU)^3}{1.327 \times 10^{20}} \\ &= 5.743 \times 10^{14} \\ \left(\frac{\tau}{2\pi}\right) &= 2.396 \times 10^7 \\ \tau &= 1.506 \times 10^8 \, s^{-1} \end{aligned}$$

The new orbital period is $1.51 \times 10^8 \, s^{-1}$ or approximately 4.77 years.
