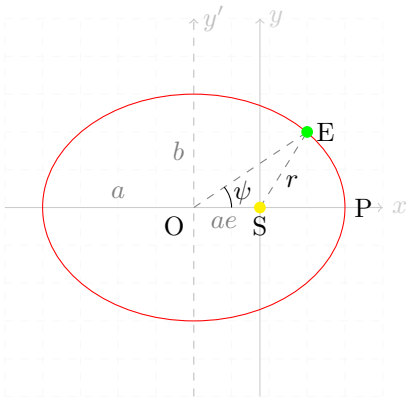
Problem

Show that the position of a planet in its elliptical orbit can be expressed, using a frame with x-axis in the direction of the *perihelion*, in terms of the angular parameter ψ by $x = a(cos\psi - e)$, $y = bsin\psi$. (ψ is the *eccentric anomaly*, while the polar angle θ is the *true anomaly*.) Show that $r = a(1 - ecos\psi)$, and that the time from the perihelion is given by $t = (\tau/2\pi)(\psi - esin\psi)$ (Kepler's equation).

Solution

Equation of ellipse centered at the origin in parametric form

$$x = acos\psi \qquad\qquad y' = bsin\psi$$



Focii are at $(\pm ae, 0)$ where $e^2 = 1 - b^2/a^2$

Hence in the sun's frame of reference

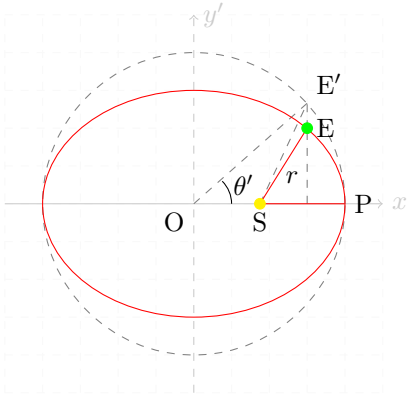
$$x = a(cos\psi - e) \qquad\qquad y = bsin\psi$$

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
&= a^2(cos\psi - e)^2 + b^2 sin^2\psi \\
&= a^2\big((cos\psi - e)^2 + (1 - e^2)sin^2\psi\big) \\
&= a^2\big((cos\psi - e)^2 + (1 - e^2)(1 - cos^2\psi)\big) \\
&= a^2\big(cos^2\psi - 2ecos\psi + e^2 + 1 - cos^2\psi - e^2 + e^2cos^2\psi\big) \\
r^2 &= a^2(1 - ecos\psi)^2 \\
r &= a(1 - ecos\psi)
\end{aligned}$$

For a circular path, area of 'sector' SPE′ $= \frac{1}{2}a^2\theta' - \frac{1}{2}(ae)(a\sin\theta')$

For an elliptical path, area of sector SPE $= \dfrac{b}{a}$(area SPE′)

Hence area SPE $= \frac{1}{2}ab\psi - \frac{1}{2}abesin\psi$



Using Kelper's second law the time taken, t, from the perihelion

$$\begin{aligned}
\frac{t}{\text{Area SPE}} &= \frac{\tau}{\pi ab} \\
t &= \frac{\tau}{\pi ab}\left(\tfrac{1}{2}ab\psi - \tfrac{1}{2}abesin\psi\right) \\
t &= \frac{\tau}{2\pi}(\psi - esin\psi)
\end{aligned}$$
