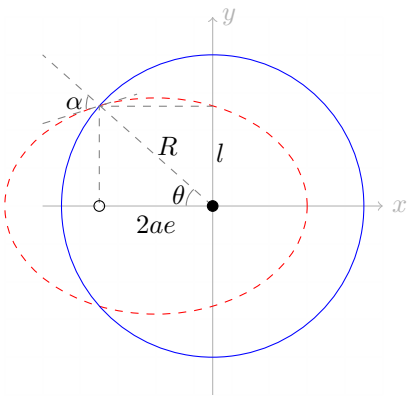
Problem

A ballistic rocket (one that moves freely under gravity after its initial launch) is fired from the surface of the Earth with velocity $v < \sqrt{Rg}$ at an angle α to the vertical. Find the equation of its orbit. Express the range $2R\theta$ in terms of parameters l and a, and hence show that to maximize the range, we should choose α so that $l = 2a - R$. Deduce the maximum range is $2R\theta$ where $sin\theta = v^2/(2Rg - v^2)$. Given that the maximum range is 3600 nautical miles, find the launch velocity and the angle at which the rocket should be launched.

Solution

The equation of the rocket is an ellipse $[r(ecos(\theta + 1) = l]$ with the Earth's centre as primary focus, hence



$$
\begin{aligned}
R(ecos\theta + 1) &= l \\
ecos\theta &= \frac{l}{R} - 1 \\
cos\theta &= \frac{l/R - 1}{e} \\
cos\theta &= \frac{l/R - 1}{\sqrt{1 - l/a}} \qquad \left(a = l/(1 - e^2)\right)
\end{aligned}
$$

$$
\text{Range} = 2R\theta = 2R\, cos^{-1}\left(\frac{l/R - 1}{\sqrt{1 - l/a}}\right)
$$

For maximum range the required trajectory is when $cos\theta = 2ae/R$ (see figure), hence

$$
\begin{aligned}
\frac{l/R - 1}{\sqrt{1 - l/a}} &= \frac{2ae}{R} \\
R - l &= 2ae^2 \\
&= 2a\left(1 - \frac{l}{a}\right) \\
R - l &= 2a - 2l \\
l &= 2a - R
\end{aligned}
$$

Orbital property of ellipse for a

$$
\begin{aligned}
a = \frac{|k|}{2|E|} &= \frac{GMm}{2(GMm/R - mv^2/2)} \\
&= \frac{R^2 g}{2Rg - v^2} \qquad (GM = R^2 g) \\
l = 2a - R &= \frac{2R^2 g}{2Rg - v^2} - R \\
&= \frac{Rv^2}{2Rg - v^2} \\
sin\theta = \frac{l}{R} &= v^2/(2Rg - v^2)
\end{aligned}
$$

A maximum range of 3600 nautical miles $\rightarrow \quad 2\theta = 60^\circ \quad sin\theta = \frac{1}{2}$

Hence

$$\begin{aligned} \frac{v^2}{2Rg - v^2} &= \frac{1}{2} \\ v^2 &= \frac{2Rg}{3} \\ &= 4.17 \times 10^7 \\ v &= 6.45 \times 10^3 ms^{-1} \end{aligned}$$

Orbital property of ellipse for b

$$\begin{aligned} b^2 = \frac{J^2}{2m|E|} &= al \\ \frac{m^2 v^2 R^2 sin^2\alpha}{2m(GMm/R - mv^2/2)} &= \frac{R^3 v^2 g}{(2Rg - v^2)^2} \\ \frac{sin^2\alpha}{2Rg - v^2} &= \frac{Rg}{(2Rg - v^2)^2} \\ sin^2\alpha &= \frac{Rg}{2Rg - v^2} \\ &= \frac{6.25 \times 10^7}{8.33 \times 10^7} = 0.75 \\ sin\alpha &= 0.866 \\ \alpha &= 60.0^\circ \end{aligned}$$

The missile should be launched with a velocity of $6.45\, kms^{-1}$ at an angle of 60.0° to the vertical.
