Problem

The orbit of an electron (charge $-e$) around a nucleus (charge Ze) is a circle of radius a in a plane perpendicular to a uniform magnetic field $\mathbf{B}$. By writing the equation of motion in a frame rotating with the electron, show that the angular velocity ω is given by one of the roots of the equation

$$m\omega^2 - eB\omega - Ze^2/4\pi\epsilon_0 a^3 = 0$$

Evaluate the two roots if $B = 10^5\,T$, $Z = 1$ and $a = 5.3 \times 10^{-11} m$. (Note: in reality a would be changed by the field.)

Solution

The electron's orbit is usually an ellipse which precesses around the nucleus, however, in this case the orbit is a circle (see §5.5 in the text - Lamour effect). It's equation of motion in the static frame is

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{Ze^2}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} - e\,\frac{d\mathbf{r}}{dt} \times \mathbf{B}$$

In the rotating frame

$$m\big(\ddot{\mathbf{r}} + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\big) = -\frac{Ze^2}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} - e\,\big((\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}\big)$$

$$\begin{aligned}
m\omega^2 a &= \frac{Ze^2}{4\pi\epsilon_0 a^2} + e\omega a B \\
m\omega^2 - eB\,\omega - \frac{Ze^2}{4\pi\epsilon_0 a^3} &= 0 \\
\omega &= \frac{eB \pm \sqrt{e^2B^2 - 4mZe^2/4\pi\epsilon_0 a^3}}{2m} \\
&= \frac{eB}{2m} \pm \sqrt{\left(\frac{eB}{2m}\right)^2 + \frac{Ze^2}{4\pi\epsilon_0 m a^3}} \\
&= \frac{eB}{2m} \pm \sqrt{\frac{Ze^2}{4\pi\epsilon_0 m a^3}} \qquad \text{(assuming } B \text{ weak)} \\
\omega &= \omega_L \pm \omega_0
\end{aligned}$$

Where $\omega_L = eB/2m$ (Lamour freqency) and $\omega_0^2 = Ze^2/4\pi\epsilon_0 m a^3$ i.e. angular velocity with no magnetic field.

$B = 10^5\,T$, $Z = 1$ and $a = 5.3 \times 10^{-11} m$

$$\begin{aligned}
\omega &= \frac{(1.6 \times 10^{-19})(10^5)}{2(9.1 \times 10^{-31})} \pm \sqrt{\frac{1 \times (1.6 \times 10^{-19})^2}{4\pi(8.85 \times 10^{-12})(9.1 \times 10^{-31})(5.3 \times 10^{-11})^3}} \\
&= 8.79 \times 10^{15} \pm \sqrt{1.70 \times 10^{33}} \\
&= 8.79 \times 10^{15} \pm 4.12 \times 10^{16} \\
\omega^+ &= 5.0 \times 10^{16}\,s^{-1} \qquad \& \qquad \omega^- = -3.2 \times 10^{16}\,s^{-1}
\end{aligned}$$

The possible angular velocities are $5.0 \times 10^{16}\,s^{-1}$ and $-3.2 \times 10^{16}\,s^{-1}$.