Problem

A projectile is launched due north from a point in colatitude θ at an angle of $\pi/4$ to the horizontal, and aimed at a target whose distance is y (small compared to the Earth's radius R). Show that if no allowance is made for the effects of the Coriolis force, the projectile will miss its target by a distance

$$x = \omega \left(\frac{2y^3}{g}\right)^{1/2} \left(cos\theta - \tfrac{1}{3} sin\theta\right)$$

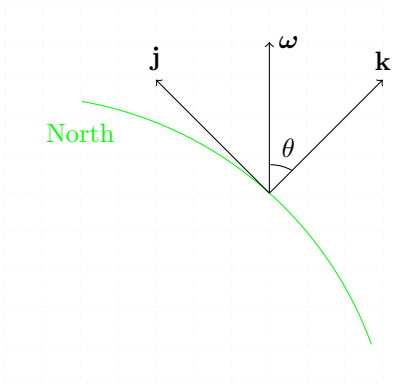
Evaluate this distance if $\theta = 45°$ and $y = 40\,km$. Why is it that the deviation is to the east near the north pole, but to the west both on the equator and near the south pole?

Solution

Choosing axes so that **i** is east, **j** is north and **k** up.

Neglecting the Coriolis and taking v as initial velocity, the equations of motion for the projectile

$$x(t) = 0 \qquad y(t) = v cos(\pi/4)\, t \qquad z(t) = v sin(\pi/4)\, t - \tfrac{1}{2} g t^2$$



Hence

$$y = \frac{vt}{\sqrt{2}} \quad \dot{y} = \frac{v}{\sqrt{2}} \qquad\qquad z = \frac{vt}{\sqrt{2}} - \tfrac{1}{2} g t^2 \quad \dot{z} = \frac{v}{\sqrt{2}} - gt$$

In the chosen co-ordinates the Coriolis force is (see §5.4 in the text)

$$-2m\,\boldsymbol{\omega} \times \dot{\mathbf{r}} = 2m\omega(\dot{y} cos\theta - \dot{z} sin\theta,\, -\dot{x} cos\theta,\, \dot{x} sin\theta)$$

Using the above approximation for $\dot{y}$ & $\dot{z}$, equation of motion for x, using appropriate initial conditions

$$\begin{aligned} m\ddot{x} &= 2m\omega\left(\dot{y} cos\theta - \dot{z} sin\theta\right) \\ \ddot{x} &= 2\omega \left(\frac{v}{\sqrt{2}} cos\theta - \left(\frac{v}{\sqrt{2}} - gt\right) sin\theta\right) \\ x &= 2\omega \left(\frac{vt^2}{2\sqrt{2}}\, cos\theta - \left(\frac{vt^2}{2\sqrt{2}} - \frac{gt^3}{6}\right) sin\theta\right) \end{aligned}$$

At the end of the flight $z = 0$ ($y << R$), hence

$$z = 0 \quad y = vt/\sqrt{2} \qquad \rightarrow \qquad t = 2v/g\sqrt{2} \qquad v = \sqrt{yg} \qquad t = \sqrt{2y/g}$$

Subs for v and t

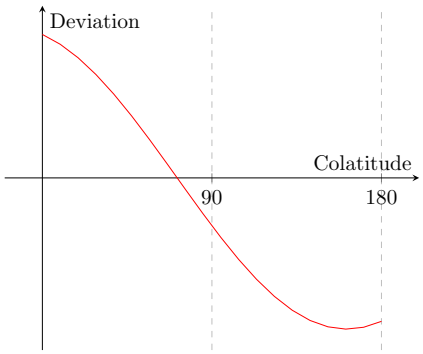
$$\begin{aligned} x &= \omega \left(\sqrt{\frac{yg}{2}} \frac{2y}{g}\, cos\theta - \left(\sqrt{\frac{yg}{2}} \frac{2y}{g} - \frac{g}{3}\left(\frac{2y}{g}\right)^{3/2}\right) sin\theta\right) \\ &= \omega \left(\frac{2y^3}{g}\right)^{1/2} \left(cos\theta - \tfrac{1}{3} sin\theta\,\right) \end{aligned}$$

For $\theta = 45°$ and $y = 40\,km$

$$\begin{aligned} x &= (7.292 \times 10^{-5}) \left(\frac{2(4 \times 10^4)^3}{9.81}\right)^{1/2} \left(cos45° - \tfrac{1}{3} sin45°\right) \\ &= (7.292 \times 10^{-5})(3.61 \times 10^6)(0.471) = 124.1 \end{aligned}$$

The projectile will miss the target by approximately $124\,m$.

A sketch of the function x for colatitudes from 0° to 180° (i.e from north to south) shows its behaviour.



Near the north pole x is +ve i.e. deviation to the east. The cosine term in the expression is dominant.

Near the equator (colatitude 90°) x is -ve i.e. deviation to the west. The sine term is dominant.

Near the south pole x is also -ve i.e. west deviation. The dominant cosine term is now -ve.