Problem

The co-ordinates (x, y, z) of a particle with respect to a uniformly rotating frame may be related to those with respect to a fixed inertial frame, (x^*, y^*, z^*), by the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cos\,\omega t & sin\,\omega t & 0 \\ -sin\,\omega t & cos\,\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}.$$

Write down the inverse relation giving (x^*, y^*, z^*) in terms of (x, y, z). By differentiating with respect to t, derive the relation between $d^2\mathbf{r}/dt^2$ and $\ddot{\mathbf{r}}$. [Hint: Note that $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$, while $d^2\mathbf{r}/dt^2$ is the vector obtained by applying the above transformation to $(\ddot{x}^*, \ddot{y}^*, \ddot{z}^*)$.]

Solution

$$\begin{vmatrix} cos\,\omega t & sin\,\omega t & 0 \\ -sin\,\omega t & cos\,\omega t & 0 \\ 0 & 0 & 1 \end{vmatrix} = cos\,\omega t(cos\,\omega t - 0) - sin\,\omega t(-sin\,\omega t - 0) + 0 = 1$$

Since the determinant is 1 the inverse matrix is simply the transpose of the original.

$$\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} cos\,\omega t & -sin\,\omega t & 0 \\ sin\,\omega t & cos\,\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

From this equation we have

$$\begin{aligned} x^* &= x cos\,\omega t - y sin\,\omega t \\ \dot{x}^* &= \dot{x} cos\,\omega t - x\omega sin\,\omega t - \dot{y} sin\,\omega t - y\omega cos\,\omega t \\ \ddot{x}^* &= \ddot{x} cos\,\omega t - 2\dot{x}\omega sin\,\omega t - x\omega^2 cos\,\omega t - \ddot{y} sin\,\omega t - 2\dot{y}\omega cos\,\omega t + y\omega^2 sin\,\omega t \end{aligned}$$

Similarly

$$\begin{aligned} \ddot{y}^* &= \ddot{x} sin\,\omega t + 2\dot{x}\omega cos\,\omega t - x\omega^2 sin\,\omega t + \ddot{y} cos\,\omega t - 2\dot{y}\omega sin\,\omega t - y\omega^2 cos\,\omega t \\ \ddot{z}^* &= \ddot{z} \end{aligned}$$

Thus applying the original transformation to these values we have

$$\begin{aligned} \frac{d^2x}{dt^2} &= \ddot{x}^* cos\,\omega t + \ddot{y}^* sin\,\omega t \\ &= \ddot{x} cos^2\,\omega t - 2\dot{x}\omega sin\,\omega t\, cos\,\omega t - x\omega^2 cos^2\,\omega t - \ddot{y} sin\ cos\,\omega t \omega t - 2\dot{y}\omega cos^2\,\omega t + y\omega^2 sin\,\omega t\, cos\,\omega t \\ &+ \ddot{x} sin^2\,\omega t + 2\dot{x}\omega sin\,\omega t\, cos\,\omega t - x\omega^2 sin^2\,\omega t + \ddot{y} sin\,\omega t\, cos\,\omega t - 2\dot{y}\omega sin^2\,\omega t - y\omega^2 sin\,\omega t\, cos\,\omega t \\ &= \ddot{x} - 2\omega\dot{y} - \omega^2 x \end{aligned}$$

Similarly

$$\frac{d^2y}{dt^2} = \ddot{y} + 2\omega\dot{x} - \omega^2 y \qquad \text{and} \qquad \frac{d^2z}{dt^2} = \ddot{z}$$

Hence

$$\frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} d^2x/dt^2 \\ d^2y/dt^2 \\ d^2z/dt^2 \end{pmatrix} = \begin{pmatrix} \ddot{x} - 2\omega\dot{y} - \omega^2 x \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y \\ \ddot{z} \end{pmatrix}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} + 2\begin{pmatrix} -\omega\dot{y} \\ \omega\dot{x} \\ 0 \end{pmatrix} + \begin{pmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{pmatrix}$$

or

$$\frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Since

$$\boldsymbol{\omega} \times \dot{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = \begin{pmatrix} -\omega\dot{y} \\ \omega\dot{x} \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} = \begin{pmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{pmatrix}$$