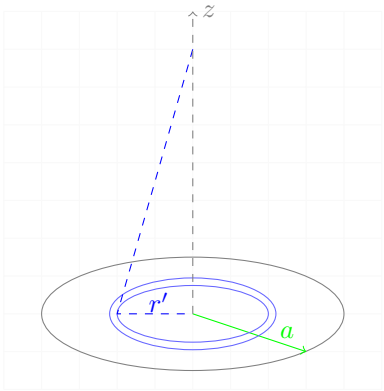
Problem

a) Find the potential and electric field at points on the axis of symmetry of a uniformly charged flat circular disc of charge q and radius a. What happens to the field if we keep the charge per unit area $\sigma = q/\pi a^2$ fixed, and let $a \to \infty$?

b) Calculate the quadruple moment of the disc and deduce the form of the field at large distances in any direction. Verify that on the axis it agrees with the exact value when r/a is sufficiently large.

Solution

a) Potential & field on the axis of symmetry



$$\begin{aligned}
\phi &= \int_0^a \frac{(2\pi r' dr')(q/\pi a^2)}{4\pi\epsilon_0 (z^2 + r'^{\,2})^{1/2}} \\
&= \frac{q}{2\pi\epsilon_0 a^2} \int_0^a \frac{r'}{(z^2 + r'^{\,2})^{1/2}}\, dr' \\
&= \frac{q}{2\pi\epsilon_0 a^2} \left[(z^2 + r'^{\,2})^{1/2} \right]_0^a \\
\phi &= \frac{q}{2\pi\epsilon_0 a^2} \left(\sqrt{z^2 + a^2} - |z| \right)
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} &= -\frac{\partial \phi}{\partial z}\, \mathbf{k} \\
&= \frac{q}{2\pi\epsilon_0 a^2} \left(\frac{z}{|z|} - \frac{z}{\sqrt{z^2 + a^2}} \right) \mathbf{k} \\
&= \frac{\sigma}{2\epsilon_0} \left(\frac{z}{|z|} - \frac{z}{\sqrt{z^2 + a^2}} \right) \mathbf{k}
\end{aligned}$$

If $a \to \infty$ the second term vanishes $\quad \mathbf{E} \quad \to \quad \dfrac{\sigma}{2\epsilon_0} \left(\dfrac{z}{|z|} \right) \mathbf{k}$

b) Quadrupole moment & potential (see §6.4 in the text)

$$\begin{aligned}
Q &= \int_0^a \rho(-r'^{\,2})(2\pi r' dr') \\
&= -\frac{2q}{a^2} \int_0^a r'^{\,3}\, dr' \\
&= -\frac{2q}{a^2} \left(\frac{a^4}{4} \right) \\
Q &= -\tfrac{1}{2} q a^2
\end{aligned}$$

$$\phi_2(\mathbf{r}) = \frac{Q}{16\pi\epsilon_0 r^3} (3cos^2\theta - 1) \qquad \text{(in spherical polars)}$$

When $r >> r'$ a series expansion in powers of $1/r$ may be used to approximate ϕ

$$\phi(\mathbf{r}) = \phi_0(\mathbf{r}) + \phi_1(\mathbf{r}) + \phi_2(\mathbf{r}) + ...$$

The origin has been chosen as the centre of charge, hence $\phi_1 = 0$ (dipole term)

$$\begin{aligned}
\phi(\mathbf{r}) &\approx \frac{q}{4\pi\epsilon_0 r} + \frac{Q}{16\pi\epsilon_0 r^3}(3cos^2\theta - 1) \\
\phi(z) &\approx \frac{q}{4\pi\epsilon_0 z} - \frac{qa^2/2}{16\pi\epsilon_0 z^3}\,(3-1) \\
&= \frac{q}{4\pi\epsilon_0 z} - \frac{qa^2}{16\pi\epsilon_0 z^3} \\
&= \frac{q}{4\pi\epsilon_0 z}\left(1 - a^2/4z^2\right)
\end{aligned}$$

Verifying from result (a) for $z >> a$

$$\begin{aligned}
\phi &= \frac{q}{2\pi\epsilon_0 a^2}\left(\sqrt{z^2+a^2} - |z|\right) \\
&= \frac{q}{2\pi\epsilon_0 a^2}\left(z(1 + a^2/z^2)^{1/2} - |z|\right) \\
&\approx \frac{qz}{2\pi\epsilon_0 a^2}\left(\frac{1}{2}\frac{a^2}{z^2} - \frac{1}{8}\frac{a^4}{z^4} + ...\right) \\
&= \frac{q}{4\pi\epsilon_0 z}\left(1 - a^2/4z^2\right)
\end{aligned}$$

Both methods yield the same result for the approximate potential on the axis when $r >> a$.
