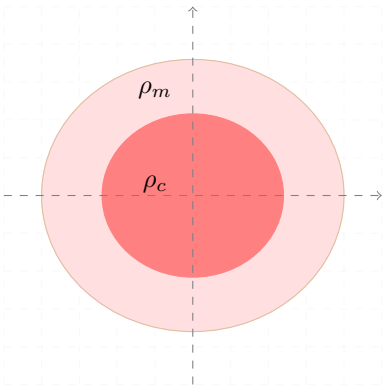
Problem

Assume that the Earth consists of a uniform core of uniform density ρ_c, surrounded by a mantle of uniform density ρ_m, and that the boundary between the two is of similar shape to the outer surface, but with a radius only three-fifths as large. Find what ratio of densities ρ_c/ρ_m is required to explain the the observed quadrupole moment. (Hint: Treat the Earth as a superposition of two ellipsoids of densities ρ_m and $\rho_c - \rho_m$.)

Solution

The Earth is approximately an oblate spheroid with oblateness ϵ defined (and measured experimentally) as

$$\epsilon = \frac{a-c}{a} = 0.0034$$



The quadrupole moment is approximately $-\frac{4}{5}Ma^2\epsilon$ for a spheroid of uniform density (see §6.5 in the text).

Using the suggested model (V as volume of Earth)

$$\begin{aligned} Q_m &= -\tfrac{4}{5}V\rho_m a^2\epsilon \\ Q_c &= -\tfrac{4}{5}\left(\frac{3}{5}\right)^3 V(\rho_c-\rho_m)\left(\frac{3a}{5}\right)^2\epsilon \\ &= -\tfrac{4}{5}V(\rho_c-\rho_m)\left(\frac{243}{3125}\right)a^2\epsilon \\ Q &= Q_m + Q_c \\ &= -\tfrac{4}{5}Va^2\left(\frac{2882\rho_m + 243\rho_c}{3125}\right)\epsilon \end{aligned}$$

Q is further defined (and also measured experimentally) in terms of the dimensionless parameter J_2. (J_l (Legendre polynomial) parametrizes the term in Φ proportional to $P_l(cos\theta)$ - see text.)

$$Q = -2Ma^2J_2 \quad \text{where} \quad J_2 = 0.0011$$

Mass of Earth

$$M = V\rho_m + \left(\frac{3}{5}\right)^3 V(\rho_c-\rho_m) = V\left(\frac{98\rho_m+27\rho_c}{125}\right)$$

Equating the two values of Q

$$\begin{aligned} -2V\left(\frac{98\rho_m+27\rho_c}{125}\right)a^2J_2 &= -\tfrac{4}{5}Va^2\left(\frac{2882\rho_m+243\rho_c}{3125}\right)\epsilon \\ 125(98\rho_m+27\rho_c)J_2 &= 2(2882\rho_m+243\rho_c)\epsilon \\ 125(98+27R)(0.0011) &= 2(2882+243R)(0.0034) \quad \text{where} \quad R=\rho_c/\rho_m \\ R &= \frac{19.60-13.48}{3.713-1.652} \\ &= \frac{6.123}{2.060} = 2.972 \end{aligned}$$

Hence for this model to explain the observed facts the ratio of densities required is $\rho_c/\rho_m \approx 3:1$.