Problem

Two small identical uniform spheres of density ρ and radius r are orbiting the Earth in a circular orbit of radius a. Given that the spheres are just touching, with their centres in line with the Earth's centre, and that the only force between them is gravitational, show that they will be pulled apart by the Earth's tidal force if a is less than $a_c = (2\rho_E/\rho)^{1/3} R_E$, where ρ_E is the mean density of the Earth and R_E its radius. (This is an illustration of the existence of the *Roche limit*.)

Solution

The Earth's *tidal* gravitational potential for a *small* satellite along the centre line to the Earth (i.e. taking $\theta = 0$ in equation 6.40)

$$g_r = \frac{2GM_E r}{a^3}$$

The tidal force towards the Earth on each satellite of mass m is approximately

$$F_r = \frac{2GM_E mr}{a^3}$$

The tidal force for the closer satellite is very slightly larger than for the other, whereas the force between them is constant whilst they are together and becomes weaker thereafter. Hence the satellites will be pulled apart when the tidal force exceeds the gravitational force between them; the *Roche limit* is when

$$\frac{2GM_E mr}{a_c^3} = \frac{Gm^2}{r^2}$$

$$a_c^3 = r^3 \left(\frac{2M_E}{m} \right)$$

Taking $M_E = \frac{4}{3}\pi R_E^3 \rho_E$ and $m = \frac{4}{3}\pi r^3 \rho$ we have

$$a_c^3 = r^3 \left(\frac{2R_E^3 \rho_E}{r^3 \rho} \right)$$

$$a_c = R_E \left(\frac{2\rho_E}{\rho} \right)^{1/3}$$

The satellites will be pulled apart when $a < a_c$.

Note: There appears to be a misprint in the question in the book which gives the Roche limit as $a_c = 2(\rho_E/\rho)^{1/3} R_E$.