Problem

Assume that the pressure p in a star with spherical symmetry is related to the density ρ by the (distinctly unrealistic) equation of state $p = \frac{1}{2}k\rho^2$, where k is a constant. Use the fluid equilibrium equation $\boldsymbol{\nabla} p = \rho \mathbf{g}^*$ to find a relation between ρ and Φ. Hence show that Poisson's equation yields

$$\frac{d^2[r\rho(r)]}{dr^2} = -\frac{4\pi G}{k}\, r\rho(r)$$

Solve this equation with the boundary conditions that ρ is finite at $r = 0$ and vanishes at the surface of the star. Hence show that the radius a of the star is determined solely by k and is independent of its mass M. Show also that $M = (4/\pi)a^3\rho(0)$.

Solution

For a star with spherical symmetry, the fluid equilibrium equation

$$\begin{aligned} \boldsymbol{\nabla} p &= \rho \mathbf{g} \\ \frac{\mathrm{d}p}{\mathrm{d}r} &= -\rho \frac{\mathrm{d}\Phi}{\mathrm{d}r} \\ k\rho \frac{\mathrm{d}\rho}{\mathrm{d}r} &= -\rho \frac{\mathrm{d}(-k\rho)}{\mathrm{d}r} \qquad\qquad (p = \tfrac{1}{2}k\rho^2) \\ \Phi &= -k\rho \end{aligned}$$

Poisson's equation for gravitational potential

$$\begin{aligned} \boldsymbol{\nabla}^2\Phi(r) &= 4\pi G\rho \\ \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) &= 4\pi G\rho \\ -\frac{k}{r^2}\left(r^2\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) &= 4\pi G\rho \\ r\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} + 2\frac{\mathrm{d}\rho}{\mathrm{d}r} &= -\frac{4\pi G}{k}\, r\rho \\ \frac{\mathrm{d}^2(r\rho)}{\mathrm{d}r^2} &= -\frac{4\pi G}{k}\, r\rho \end{aligned}$$

A general solution to this equation is

$$r\rho = A\, cos\left(\sqrt{\frac{4\pi G}{k}}\, r - \alpha\right)$$

In order to satisfy the boundary conditions, the solution is split

a) $r = 0$ and ρ finite

$$\begin{aligned} \rho(0) &= A \lim_{r\to 0} \frac{cos\left(\sqrt{\frac{4\pi G}{k}}\, r - \alpha\right)}{r} \qquad\qquad \rightarrow \quad \alpha = \pi/2 \\ &= A \lim_{r\to 0} \frac{sin\left(\sqrt{\frac{4\pi G}{k}}\, r\right)}{r} \\ \rho(0) &= A\sqrt{\frac{4\pi G}{k}} \end{aligned}$$

b) $0 < r \leq a$

$$\rho(r) = \frac{A}{r} cos\left(\sqrt{\frac{4\pi G}{k}}\, r - \frac{\pi}{2}\right) \quad \text{or} \quad \frac{A}{r} sin\left(\sqrt{\frac{4\pi G}{k}}\, r\right)$$

c) $r = a$ and $\rho = 0$

$$\begin{aligned}
\rho(a) = 0 &= \frac{A}{a} cos\left(\sqrt{\frac{4\pi G}{k}}\, a - \frac{\pi}{2}\right) \\
\sqrt{\frac{4\pi G}{k}}\, a - \frac{\pi}{2} &= \frac{\pi}{2} \\
a &= \pi\sqrt{\frac{k}{4\pi G}} \\
a &= \sqrt{\frac{\pi k}{4G}}
\end{aligned}$$

The radius of the star a is determined solely by k, and not its mass M.

Mass of star

$$\begin{aligned}
M &= \int_0^a \rho(r)(4\pi r^2\, dr) \\
&= 4\pi A \int_0^a r sin\left(\sqrt{\frac{4\pi G}{k}}\, r\right) dr \\
&= 4\pi A \left[\frac{-r cos\left(\sqrt{\frac{4\pi G}{k}}\, r\right)}{\sqrt{\frac{4\pi G}{k}}} + \frac{sin\left(\sqrt{\frac{4\pi G}{k}}\, r\right)}{\sqrt{\frac{4\pi G}{k}}}\right]_0^{\sqrt{\pi k/4G}} \\
&= 4\pi A \left(\frac{-a cos\pi}{\sqrt{4\pi G/k}}\right) \\
&= 4\pi \left(\frac{\rho(0)}{\sqrt{4\pi G/k}}\right)\left(\frac{a}{\sqrt{4\pi G/k}}\right) \\
&= \frac{\rho(0)a}{G}\, k \\
&= \left(\frac{\rho(0)a}{G}\right)\left(\frac{4Ga^2}{\pi}\right) \\
&= \frac{4}{\pi} a^3 \rho(0)
\end{aligned}$$

The mass of the star is $4a^3\rho(0)/\pi$.

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* This was derived in Problem 23